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## Problem 1: Scalar- and vector operators (Oral)

**Learning objective**

Rotations in space are described on the Hilbert space by angular momentum operators. Some operators on the Hilbert space transform like scalars or vectors under rotations. Knowledge about these so called scalar- and vector operators is extremely useful to decide whether a given Hamiltonian is rotationally invariant and to evaluate commutators without cumbersome calculations. Here you derive some important properties to identify scalar- and vector operators.

Consider rotations in three dimensions about an axis $\hat{\boldsymbol{\omega}} = \boldsymbol{\omega}/\omega$ with angle $\omega$. The angular momentum operator $\mathbf{L} = \mathbf{r} \wedge \mathbf{p}$ is the infinitesimal generator of rotations and $U_{\boldsymbol{\omega}} = \exp\left(-i\omega\, \mathbf{L} \cdot \hat{\boldsymbol{\omega}}/\hbar\right)$ represents the rotations on the Hilbert space of wave functions, i.e., $|\Psi_{\boldsymbol{\omega}}\rangle = U_{\boldsymbol{\omega}}\,|\Psi\rangle$ describes the wave function of the rotated system.

By definition, a *scalar* operator $S$ is invariant under rotations,

$$U_{\boldsymbol{\omega}}^{\dagger}\, S\, U_{\boldsymbol{\omega}} = S\,, \tag{1}$$

and a *vector* operator $\mathbf{V} = (V_x, V_y, V_z)^T$ transforms like

$$U_{\boldsymbol{\omega}}^{\dagger}\, \mathbf{V}\, U_{\boldsymbol{\omega}} = \mathcal{R}_{\boldsymbol{\omega}} \mathbf{V}\,, \tag{2}$$

where $\mathcal{R}_{\boldsymbol{\omega}}$ denotes the usual rotation matrix for vectors in three dimensions.

a) Show that for a scalar operator $S$ it is $[\mathbf{L}, S] = 0$ (shorthand for $[L_\alpha, S] = 0$ for $\alpha = x, y, z$).

b) Show that for a vector operator $\mathbf{V}$ it is $[L_i, V_j] = i\hbar\varepsilon_{ijk}V_k$.
**Hint:** Use the representation $(\mathcal{R}_{\boldsymbol{\omega}})_{ij} = [1 - \cos(\omega)]\hat{\omega}_i\hat{\omega}_j + \cos(\omega)\,\delta_{ij} - \sin(\omega)\,\varepsilon_{ijk}\hat{\omega}_k$ for the rotation matrix and linearize (2) for small $\omega$.

c) Use that $\mathbf{r}$ and $\mathbf{p}$ are vector operators to show that $\mathbf{L}$ is also a vector operator.
**Hint:** Consider the components of $U_{\boldsymbol{\omega}}^{\dagger}\, \mathbf{r} \wedge \mathbf{p}\, U_{\boldsymbol{\omega}}$ and show that $U_{\boldsymbol{\omega}}^{\dagger}\, \mathbf{r} \wedge \mathbf{p}\, U_{\boldsymbol{\omega}} = U_{\boldsymbol{\omega}}^{\dagger}\, \mathbf{r}\, U_{\boldsymbol{\omega}} \wedge U_{\boldsymbol{\omega}}^{\dagger}\, \mathbf{p}\, U_{\boldsymbol{\omega}}$.

d) Use that $\mathbf{r}$ and $\mathbf{p}$ are vector operators to show that $[\mathbf{L}, \mathbf{p} \cdot \mathbf{r}] = 0$ first by explicitely calculating the commutator, and second by showing that $\mathbf{p} \cdot \mathbf{r}$ is a scalar operator.

### Problem 2: Spin rotations (Written)

**Learning objective**

Here you derive a useful formula for spin-$\frac{1}{2}$ rotations. The application to states yields an explicit expression for their transformation under rotations and reveals a peculiarity of spin (in contrast to orbital angular momentum).

a) Show that the spin-$\frac{1}{2}$ representation of rotations about the axis $\hat{\boldsymbol{\omega}} = \boldsymbol{\omega}/\omega$ with angle $\omega = |\boldsymbol{\omega}|$ evaluates to

$$\exp\left(-\frac{i}{2}\boldsymbol{\omega}\cdot\boldsymbol{\sigma}\right) = \mathbb{1}\cos\frac{\omega}{2} - i\hat{\boldsymbol{\omega}}\cdot\boldsymbol{\sigma}\,\sin\frac{\omega}{2} \tag{3}$$

where $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)^T$ are the Pauli matrices.

**Hint:** Use that $\sigma_i\sigma_j = \delta_{ij}\mathbb{1} + i\varepsilon_{ijk}\sigma_k$.

b) The rotation of a spin-$\frac{1}{2}$ particle is described by $U_{\boldsymbol{\omega}} = \exp\left(-i\omega\,\mathbf{S}\cdot\hat{\boldsymbol{\omega}}/\hbar\right)$ with the spin operator $S_i = \frac{\hbar}{2}\sigma_i$ (c.f. Problem 1).

Evaluate $|\Psi_{\boldsymbol{\omega}}\rangle = U_{\boldsymbol{\omega}}\,|+\rangle$ explicitely for $|+\rangle = (|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$ and $\hat{\boldsymbol{\omega}} = (1,0,0)^T$. What happens for a full rotation with $\omega = 2\pi$ and is this a problem from the physical perspective?

## Problem 3: Clebsch-Gordan coefficients and spin-orbit coupling (Oral)

**Learning objective**

In this problem you apply the angular momentum addition theorem. As an important use case, we consider the spin-orbit coupling in the hydrogen atom, which is the leading relativistic correction (see end of the course). In particular, you practice the construction of the new basis states for a fixed total angular momentum.

The spin-orbit coupling between the electron's spin **S** and the orbital angular momentum **L** for a hydrogen atom is given by the Hamiltonian

$$H_{\mathrm{LS}} = f(r)\,\mathbf{L}\cdot\mathbf{S} = f(r)\sum_{\alpha=x,y,z} L_\alpha \otimes S_\alpha\,, \tag{4}$$

where $f(r) = e^2/2m_e^2c^2r^3$. The spin-orbit coupling can be seen as a perturbation to the non-relativistic Hamiltonian $H_0 = \mathbf{P}^2/2m - e^2/r$ of the hydrogen atom.

a) Define the total angular momentum operator as

$$\mathbf{J} = \mathbf{L} + \mathbf{S} = \mathbf{L}\otimes\mathbb{1} + \mathbb{1}\otimes\mathbf{S} \tag{5}$$

and show that $\mathbf{J}^2$ and $J_z$ commute both with $H_0$ and $H_{\mathrm{LS}}$.

b) Consider the subspace with orbital angular momentum $\ell$ and spin $s$. We can write the eigenstates $|j,m\rangle$ of $\mathbf{J}^2$ and $J_z$ as linear combinations of $L_z$- and $S_z$-eigenstates $|m_\ell, m_s\rangle = |\ell, m_\ell\rangle \otimes |s, m_s\rangle$,

$$|j,m\rangle = \sum_{m_\ell,m_s} c(m_\ell, m_s; j, m)\,|m_\ell, m_s\rangle\,. \tag{6}$$

The coefficients $c$ are called *Clebsch-Gordan coefficients*. Due to their ubiquity in quantum physics there are comprehensive tables available, e.g.,

`http://pdg.lbl.gov/2011/reviews/rpp2011-rev-clebsch-gordan-coefs.pdf`.

Use this table to write down the change of basis (6) in the subspace with $\ell = 1$ and $s = 1/2$ explicitely.

c) Derive the Clebsch-Gordan coefficients in b) by hand.
**Hint:** Start with the *stretched state* $|j = 3/2, m_j = 3/2\rangle$ and use the ladder operator $J_- = J_x - iJ_y$ which acts as

$$J_-\,|j,m\rangle = \hbar\sqrt{j(j+1) - m(m-1)}\,|j, m-1\rangle\,. \tag{7}$$