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Problem 1: Properties of bosonic operators (Oral)

Learning objective

The first part of this problem reviews properties of bosonic creation and annihilation operators in a single-particle Fock space, whereas the last part generalizes this concept to many-particle systems.

In order to solve the harmonic oscillator, one can introduce bosonic creation and annihilation operators satisfying the commutation relation

$$[b, b^\dagger] = 1\,. \tag{1}$$

The occupation number operator is given by $\hat{n} = b^\dagger b$ with eigenstates $|n\rangle$ and eigenvalues n.

a) Using (1), show that $b\,|n\rangle$ and $b^\dagger\,|n\rangle$ are eigenstates of $\hat{n}$.

b) Prove the following relations:

$$b\,|n\rangle = \sqrt{n}\,|n-1\rangle\ , \tag{2}$$

$$b^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle\ . \tag{3}$$

c) Show that there has to be a state $|G\rangle$ with $b\,|G\rangle = 0$ and prove that n is an integer.
Hint: Use the fact that there are no states with negative norm.

d) Using the commutation relations of operators in bosonic Fock space,

$$[b_i, b_j^\dagger] = \delta_{ij}, \quad [b_i, b_j] = 0, \quad [b_i^\dagger, b_j^\dagger] = 0 \tag{4}$$

and the states

$$|n_1, \ldots, n_i, \ldots\rangle = \ldots \frac{(b_i^\dagger)^{n_i}}{\sqrt{n_i!}} \ldots \frac{(b_1)^\dagger}{\sqrt{n_1!}}\,|0\rangle\,, \tag{5}$$

prove the following relations:

$$b_i\,|n_1, \ldots, n_i, \ldots\rangle = \sqrt{n_i}\,|n_1, \ldots, n_i - 1, \ldots\rangle\ , \tag{6}$$

$$b_i^\dagger\,|n_1, \ldots, n_i, \ldots\rangle = \sqrt{n_i + 1}\,|n_1, \ldots, n_i + 1, \ldots\rangle\ . \tag{7}$$

e) Show that the states (5) build an orthonormal basis. Define the total number operator $\hat{n} = \sum \hat{n}_i = \sum b_i^\dagger b_i$. Now take a Hilbert-space of m-modes, i.e., the basis states of the Fock space are

$$|n_1, \ldots, n_m\rangle\,.$$

How many states $|\Psi\rangle$ exists with a given particle number $N = \hat{n}\,|\Psi\rangle$.

Problem 2: Properties of fermionic operators (Oral)

Learning objective

The second part of this problem reviews properties of fermionic creation and annihilation operators in a single-particle Fock space, whereas the last part generalizes this concept to many-particle systems.

Now we introduce fermionic operators, which fulfil the anti commutation relation

$$\{a, a^\dagger\} = 1\,. \tag{8}$$

The occupation number operator is given by $\hat{n} = a^\dagger a$ with eigenstates $|n\rangle$ and eigenvalues n.

a) Using (8), show that $a\,|n\rangle$ and $a^\dagger\,|n\rangle$ are eigenstates of $\hat{n}$.

b) Prove the following relations:

$$a\,|n\rangle = \sqrt{n}\,|1-n\rangle = a^\dagger\,|n\rangle\,. \tag{9}$$

c) Show that there has to be a state $|G\rangle$ with $a\,|G\rangle = 0$ and a state $|H\rangle$ with $a^\dagger\,|H\rangle = 0$. Further show, that n is an integer.
Hint: Use the fact that there are no states with negative norm.

d) Using the anti commutation relations of operators in fermionic Fock space,

$$[a_i, a_j^\dagger] = \delta_{ij}, \quad [a_i, a_j] = 0, \quad [a_i^\dagger, a_j^\dagger] = 0 \tag{10}$$

and the states

$$|n_1, \ldots, n_i, \ldots\rangle = \ldots (a_i^\dagger)^{n_i} \ldots (a_1)^\dagger\,|0\rangle\,, \tag{11}$$

prove the following relations:

$$a_i\,|n_1, \ldots, n_i, \ldots\rangle = \delta_{n_i,1}(-1)^{S_i}\,|n_1, \ldots, n_i - 1, \ldots\rangle\,, \tag{12}$$
$$a_i^\dagger\,|n_1, \ldots, n_i, \ldots\rangle = \delta_{n_i,0}(-1)^{S_i}\,|n_1, \ldots, n_i + 1, \ldots\rangle\,, \tag{13}$$

where $S_i = n_\infty + \cdots + n_{i+1}$

e) Show that the states (11) build an orthonormal basis. Define the total number operator $\hat{n} = \sum \hat{n}_i = \sum a_i^\dagger a_i$. Now take a Hilbert-space of m-modes, i.e., the basis states of the Fock space are

$$|n_1, \ldots, n_m\rangle\,.$$

How many states $|\Psi\rangle$ exists with a given particle number $N = \hat{n}\,|\Psi\rangle$.

Problem 3: Expectation values of bosonic and fermionic operators (written)

Learning objective

Here you will take the basis states of the bosonic and fermionic Fock space and practice the calculation of expectation values of many-particle systems.

In the following we will work with the operators

$$\begin{aligned} A &= c_i^\dagger c_i & B &= c_i^\dagger c_i c_j^\dagger c_j \\ C &= c_i^\dagger c_j^\dagger c_j c_i & D &= c_i^\dagger c_j^\dagger c_i c_j . \end{aligned}$$

a) Show that these operators are self-adjoint. Consider the case $c_i = b_i$ and $c_i = a_i$ separately.

b) Calculate the expectation value of the operators A, B, C and D, taking the states (5), with $c_i = b_i$, and (11), with $c_i = a_i$.

c) Finally, determine the matrix element

$$\langle n_1, \ldots, n_i, \ldots | \, c_i^\dagger c_j + c_j^\dagger c_i \, | n_1, \ldots, n_i, \ldots \rangle \, ,$$

again both for the bosonic and fermionic Fock space and operator algebra.