Problem 1: Properties of bosonic operators (Oral)

Learning objective
The first part of this problem reviews properties of bosonic creation and annihilation operators in a single-particle Fock space, whereas the last part generalizes this concept to many-particle systems.

In order to solve the harmonic oscillator, one can introduce bosonic creation and annihilation operators satisfying the commutation relation
\[ [b, b^\dagger] = 1. \] (1)

The occupation number operator is given by \( \hat{n} = b^\dagger b \) with eigenstates \( |n\rangle \) and eigenvalues \( n \).

a) Using (1), show that \( b |n\rangle \) and \( b^\dagger |n\rangle \) are eigenstates of \( \hat{n} \).

b) Prove the following relations:
\[ b |n\rangle = \sqrt{n} |n - 1\rangle , \] (2)
\[ b^\dagger |n\rangle = \sqrt{n + 1} |n + 1\rangle . \] (3)

c) Show that there has to be a state \( |G\rangle \) with \( b |G\rangle = 0 \) and prove that \( n \) is an integer.

Hint: Use the fact that there are no states with negative norm.

d) Using the commutation relations of operators in bosonic Fock space,
\[ [b_i, b_j^\dagger] = \delta_{ij}, \quad [b_i, b_j] = 0, \quad [b_i^\dagger, b_j^\dagger] = 0 \] (4)

and the states
\[ |n_1, \ldots, n_i, \ldots\rangle = \ldots \frac{(b_i^\dagger)^{n_i}}{\sqrt{n_i!}} \ldots \frac{(b_1^\dagger)^{n_1}}{\sqrt{n_1!}} |0\rangle , \] (5)

prove the following relations:
\[ b_i |n_1, \ldots, n_i, \ldots\rangle = \sqrt{n_i} |n_1, \ldots, n_i - 1, \ldots\rangle , \] (6)
\[ b_i^\dagger |n_1, \ldots, n_i, \ldots\rangle = \sqrt{n_i + 1} |n_1, \ldots, n_i + 1, \ldots\rangle . \] (7)

e) Show that the states (5) build an orthonormal basis. Define the total number operator \( \hat{n} = \sum \hat{n}_i = \sum b_i^\dagger b_i \). Now take a Hilbert-space of \( m \)-modes, i.e., the basis states of the Fock space are
\[ |n_1, \ldots, n_m\rangle . \]

How many states \( |\Psi\rangle \) exists with a given particle number \( N = \hat{n} |\Psi\rangle \).
Problem 2: Properties of fermionic operators (Oral)

Learning objective
The second part of this problem reviews properties of fermionic creation and annihilation operators in a single-particle Fock space, whereas the last part generalizes this concept to many-particle systems.

Now we introduce fermionic operators, which fulfill the anti commutation relation
\[
\{ a, a^\dagger \} = 1. \tag{8}
\]

The occupation number operator is given by \( \hat{n} = a^\dagger a \) with eigenstates \( |n\rangle \) and eigenvalues \( n \).

a) Using (8), show that \( a |n\rangle \) and \( a^\dagger |n\rangle \) are eigenstates of \( \hat{n} \).

b) Prove the following relations:
\[
a |n\rangle = \sqrt{n} |1 - n\rangle = a^\dagger |n\rangle. \tag{9}
\]

c) Show that there has to be a state \( |G\rangle \) with \( a |G\rangle = 0 \) and a state \( |H\rangle \) with \( a^\dagger |H\rangle = 0 \). Further show, that \( n \) is an integer.

Hint: Use the fact that there are no states with negative norm.

d) Using the anti commutation relations of operators in fermionic Fock space,
\[
[a_i, a_j^\dagger] = \delta_{ij}, \quad [a_i, a_j] = 0, \quad [a_i^\dagger, a_j^\dagger] = 0 \tag{10}
\]

and the states
\[
|n_1, \ldots, n_i, \ldots\rangle = \ldots (a_i^\dagger)^{n_i} \ldots (a_1^\dagger) |0\rangle, \tag{11}
\]

prove the following relations:
\[
a_i |n_1, \ldots, n_i, \ldots\rangle = \delta_{n_i,1} (-1)^{S_i} |n_1, \ldots, n_i - 1, \ldots\rangle, \tag{12}
\]
\[
a_i^\dagger |n_1, \ldots, n_i, \ldots\rangle = \delta_{n_i,0} (-1)^{S_i} |n_1, \ldots, n_i + 1, \ldots\rangle, \tag{13}
\]

where \( S_i = n_\infty + \cdots + n_{i+1} \)

e) Show that the states (11) build an orthonormal basis. Define the total number operator \( \hat{n} = \sum \hat{n}_i = \sum a_i^\dagger a_i \). Now take a Hilbert-space of \( m \)-modes, i.e., the basis states of the Fock space are
\[
|n_1, \ldots, n_m\rangle.
\]

How many states \( |\Psi\rangle \) exists with a given particle number \( N = \hat{n} |\Psi\rangle \).
Problem 3: Expectation values of bosonic and fermionic operators (written)

**Learning objective**

Here you will take the basis states of the bosonic and fermionic Fock space and practice the calculation of expectation values of many-particle systems.

In the following we will work with the operators

\[ A = c_i^\dagger c_i \quad B = c_i^\dagger c_i c_j^\dagger c_j \]
\[ C = c_i^\dagger c_j^\dagger c_j c_i \quad D = c_i^\dagger c_j^\dagger c_j c_i. \]

a) Show that these operators are self-adjoint. Consider the case \( c_i = b_i \) and \( c_i = a_i \) separately.

b) Calculate the expectation value of the operators \( A, B, C \) and \( D \), taking the states (5), with \( c_i = b_i \),

and (11), with \( c_i = a_i \).

c) Finally, determine the matrix element

\[ \langle n_1, \ldots, n_i, \ldots | c_i^\dagger c_j + c_j^\dagger c_i | n_1, \ldots, n_i, \ldots \rangle, \]

again both for the bosonic and fermionic Fock space and operator algebra.