Problem 1: The pair correlation function of a Fermi Sea (Written)

Learning objective
In this problem, you apply your knowledge of calculating expectation values of fermionic operators in second quantization to determine the pair correlation function of non-interacting fermions in free space.

Consider a gas of $N$ identical non-interacting fermions with spin $1/2$ in free space. The non-local density operator $G_\sigma(r - r')$ is given by

$$G_\sigma(r - r') = \langle \Phi_0 | \Psi_\sigma^\dagger(r) \Psi_\sigma(r') | \Phi_0 \rangle$$

and the pair correlation function $g_{\sigma\sigma'}(r - r')$ is defined as

$$g_{\sigma\sigma'}(r - r') = \frac{\langle \Phi_0 | \Psi_\sigma^\dagger(r) \Psi_\sigma^\dagger(r') \Psi_\sigma(r') \Psi_\sigma(r) | \Phi_0 \rangle}{(G_\sigma(0)G_{\sigma'}(0))},$$

where $|\Phi_0\rangle$ is the Fermi sea with a total density $n = n_\uparrow + n_\downarrow$ and $n_\uparrow = n_\downarrow$. The pair correlation function describes the conditional probability of finding an electron at the position $r'$ in the spin state $\sigma'$, when we know that the second electron is at the position $r$ in the spin state $\sigma$.

a) Express the field operators in the natural basis, that is,

$$\Psi_\sigma(r) = \frac{1}{\sqrt{V}} \sum_p c_p^\dagger \sigma c_p \sigma,$$

where $c_p^\dagger \sigma$ and $c_p \sigma$ are the creation and annihilation operators of a fermion with momentum $p$ and spin $\sigma$, respectively. Use this to determine $G_\sigma(r)$.

b) In the course of calculating the pair correlation function, expectation values of the form

$$\langle \Phi_0 | c_{p\sigma}^\dagger c_{q\sigma'}^\dagger c_{q'\sigma'} c_{p'\sigma} | \Phi_0 \rangle.$$

occur. Compute these expectation values explicitly. What conditions on $p, p', q, q'$ and $\sigma, \sigma'$ have to be satisfied so that the amplitudes are non-zero?

c) Next, consider the case $\sigma \neq \sigma'$ and use the above results to calculate explicitly the pair correlation function.

d) Finally, consider the interesting case of $\sigma = \sigma'$ and determine the pair-correlation function. Sketch the result.
Problem 2: Commutator of the electric field (Oral)

Learning objective
This problem deals with the quantized electric field. You calculate its commutator and show that it preserves causality.

In this problem, we calculate the commutator of the electric field, \([E_i(\vec{r}, t), E_j(\vec{r}', t')]\).

a) In a first step, calculate the commutator \([A_i(\vec{r}, t), A_j(\vec{r}', t')]\).

Start by decomposing the vector potential \(A(\vec{r}, t)\) into its normal modes

\[
A(\vec{r}, t) = \sum_{k,\lambda} \left( \frac{\hbar c^2}{V \omega_k} \right)^{1/2} \left( \hat{a}_{k,\lambda} \epsilon(\vec{k}, \lambda) e^{i(\vec{k} \cdot \vec{r} - \omega_k t) + \text{h.c.}} \right),
\]

where \(\vec{r}\) is the spatial coordinate, \(t\) the time, \(\vec{k}\) the wave vector, \(V\) quantization volume, \(\hbar\) Planck’s constant, \(c\) the speed of light, \(\omega_k = c |\vec{k}|\), \(\hat{a}_{k,\lambda}\) the annihilation operator for wave number \(k\) and polarization \(\lambda\) and \(\epsilon\) is the vector of polarization. In addition, use the completeness relation of the polarization vectors,

\[
\sum_\lambda \epsilon_i(\vec{k}, \lambda) \epsilon^*_j(\vec{k}, \lambda) = \delta_{ij} - \frac{k_i k_j}{k^2},
\]

in order to bring the commutator into the following form

\[
[A_i(\vec{r}, t), A_j(\vec{r}', t')] = \partial_{ij} K(\xi, \tau),
\]

with \(\xi \equiv \vec{r} - \vec{r}'\) and \(\tau \equiv t - t'\), where \(K(\xi, \tau)\) has to be determined. The differential operator \(\partial_{ij}\) is defined as

\[
\partial_{ij} \equiv \frac{\partial^2}{\partial^2 \tau} \delta_{ij} - \partial_{\xi_i} \partial_{\xi_j}.
\]

b) Next, write the commutator for the electric field \(E\) in the following form

\[
[E_i(\vec{r}, t), E_j(\vec{r}', t')] = -\frac{1}{c^2} \frac{\partial^2}{\partial \tau^2} [A_i(\vec{r}, t), A_j(\vec{r}', t')].
\]

c) Finally, transform the commutator into a form which is proportional to \(\delta(\xi^2 - c^2 \tau^2)\). Discuss the physical concept behind this solution.

\textit{Hint:} It is not required to evaluate the derivatives \(\partial_{ij}\), it is sufficient to perform the integration over \(\vec{k}\). This can be done by replacing the summation \(\sum_k \to \int \frac{d^3k}{(2\pi)^3}\) by an integral.