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Problem 1: The pair correlation function of a Fermi Sea (Written)

Learning objective

In this problem, you apply your knowledge of calculating expectation values of fermionic operators in second quantization to determine the pair correlation function of non-interacting fermions in free space.

Consider a gas of N identical non-interacting fermions with spin $1/2$ in free space. The non-local density operator $G_s(\mathbf{r}-\mathbf{r}')$ is given by

$$G_\sigma(\mathbf{r}-\mathbf{r}') = \langle \Phi_0 | \Psi^\dagger_\sigma(\mathbf{r})\Psi_\sigma(\mathbf{r}') |\Phi_0\rangle$$

and the pair correlation function $g_{\sigma\sigma'}(\mathbf{r}-\mathbf{r}')$ is defined as

$$g_{\sigma\sigma'}(\mathbf{r}-\mathbf{r}') = \langle \Phi_0 | \Psi^\dagger_\sigma(\mathbf{r})\Psi^\dagger_{\sigma'}(\mathbf{r}')\Psi_{\sigma'}(\mathbf{r}')\Psi_\sigma(\mathbf{r}) |\Phi_0\rangle / (G_\sigma(\mathbf{0})G_{\sigma'}(\mathbf{0})), \tag{1}$$

where $|\Phi_0\rangle$ is the Fermi sea with a total density $n = n_\uparrow + n_\downarrow$ and $n_\uparrow = n_\downarrow$. The pair correlation function describes the conditional probability of finding an electron at the position $\mathbf{r}'$ in the spin state σ', when we know that the second electron is at the position $\mathbf{r}$ in the spin state σ.

a) Express the field operators in the natural basis, that is,

$$\Psi_\sigma(\mathbf{r}) = \frac{1}{\sqrt{V}}\sum_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{r}} c_{\mathbf{p}\sigma}, \tag{2}$$

where $c^\dagger_{\mathbf{p}\sigma}$ and $c_{\mathbf{p}\sigma}$ are the creation and annihilation operators of a fermion with momentum $\mathbf{p}$ and spin σ, respectively. Use this to determine $G_\sigma(\mathbf{r})$.

b) In the course of calculating the pair correlation function, expectation values of the form

$$\langle \Phi_0 | c^\dagger_{\mathbf{p}\sigma} c^\dagger_{\mathbf{q}\sigma'} c_{\mathbf{q}'\sigma'} c_{\mathbf{p}'\sigma} |\Phi_0\rangle . \tag{3}$$

occur. Compute these expectation values explicitly. What conditions on p, p', q, q' and σ, σ' have to be satisfied so that the amplitudes are non-zero?

c) Next, consider the case $\sigma \neq \sigma'$ and use the above results to calculate explicitly the pair correlation function.

d) Finally, consider the interesting case of $\sigma = \sigma'$ and determine the pair-correlation function. Sketch the result.

Problem 2: Commutator of the electric field (Oral)

> **Learning objective**
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> This problem deals with the quantized electric field. You calculate its commutator and show that it preserves causality.

In this problem, we calculate the commutator of the electric field, $[E_i(\mathbf{r}, t)\,, E_j(\mathbf{r}', t')]$.

a) In a first step, calculate the commutator $[A_i(\mathbf{r}, t)\,, A_j(\mathbf{r}', t')]$.

Start by decomposing the vector potential $\mathbf{A}(\mathbf{r}, t)$ into its normal modes

$$\mathbf{A}(\mathbf{r}, t) = \sum_{\mathbf{k},\lambda} \left(\frac{hc^2}{V\omega_{\mathbf{k}}}\right)^{1/2} \left(\hat{a}_{\mathbf{k},\lambda}\boldsymbol{\epsilon}(\mathbf{k}, \lambda)e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}t} + \text{h.c.}\right), \tag{4}$$

where $\mathbf{r}$ is the spatial coordinate, t the time, $\mathbf{k}$ the wave vector, V quantization volume, h Planck's constant, c the speed of light, $\omega_{\mathbf{k}} = c|\mathbf{k}|$, $a_{\mathbf{k},\lambda}$ the annihilation operator for wave number $\mathbf{k}$ and polarization λ and $\boldsymbol{\epsilon}$ is the vector of polarization. In addition, use the completeness relation of the polarization vectors,

$$\sum_{\lambda} \epsilon_i(\mathbf{k}, \lambda)\epsilon_j^*(\mathbf{k}, \lambda) = \delta_{ij} - \frac{k_i k_j}{k^2}, \tag{5}$$

in order to bring the commutator into the following form

$$[A_i(\mathbf{r}, t), A_j(\mathbf{r}', t')] = \partial_{ij} K(\boldsymbol{\xi}, \tau), \tag{6}$$

with $\boldsymbol{\xi} \equiv \mathbf{r} - \mathbf{r}'$ and $\tau \equiv t - t'$, where $K(\boldsymbol{\xi}, \tau)$ has to be determined. The differential operator ∂_{ij} is defined as

$$\partial_{ij} \equiv \frac{\partial_\tau^2}{c^2}\delta_{ij} - \partial_{\xi_i}\partial_{\xi_j}. \tag{7}$$

b) Next, write the commutator for the electric field $\mathbf{E}$ in the following form

$$[E_i(\mathbf{r}, t), E_j(\mathbf{r}', t')] = -\frac{1}{c^2}\frac{\partial^2}{\partial\tau^2}[A_i(\mathbf{r}, t), A_j(\mathbf{r}', t')]. \tag{8}$$

c) Finally, transform the commutator into a form which is proportional to $\delta(\xi^2 - c^2\,\tau^2)$. Discuss the physical concept behind this solution.

Hint: It is not required to evaluate the derivatives ∂_{ij}, it is sufficient to perform the integration over $\mathbf{k}$. This can be done by replacing the summation $\frac{1}{V}\sum_{\mathbf{k}} \to \int \frac{d^3k}{(2\pi)^3}$ by an integral.