Problem 1: Commutator of the electric field — again © (Oral)

Learning objective
This problem deals with the quantized electric field. You calculate its commutator and show that it preserves causality.

In this problem, we calculate the commutator of the electric field, \([E_i(r, t), E_j(r', t')]\).

a) In a first step, calculate the commutator \([A_i(r, t), A_j(r', t')]\) of the vector potential.

Start by decomposing the vector potential \(A(r, t)\) into its normal modes

\[ A(r, t) = \sum_{k,\lambda} \left( \frac{\hbar c^2}{V\omega_k} \right)^{1/2} \left( a_{k,\lambda} \epsilon(k, \lambda) e^{i(k \cdot r - \omega_k t)} + \text{h.c.} \right), \]

where \(r\) is the spatial coordinate, \(t\) the time, \(k\) the wave vector, \(V\) the quantization volume, \(\hbar\) Planck’s constant, \(c\) the speed of light, and \(\omega_k = c|k|\). \(a_{k,\lambda}\) denotes the annihilation operator for wave number \(k\) and polarization \(\lambda\), \(\epsilon\) is the vector of polarization.

In addition, use the completeness relation of the polarization vectors,

\[ \sum_{\lambda} \epsilon_i(k, \lambda) \epsilon_j^\ast(k, \lambda) = \delta_{ij} - \frac{k_i k_j}{k^2}, \]

in order to bring the commutator into the following form:

\[ [A_i(r, t), A_j(r', t')] = \partial_{ij} K(\xi, \tau), \]

with \(\xi \equiv r - r'\) and \(\tau \equiv t - t'\), where \(K(\xi, \tau)\) has to be determined.

The differential operator \(\partial_{ij}\) is defined as

\[ \partial_{ij} \equiv \frac{\partial^2}{c^2} \delta_{ij} - \partial_{\xi_i} \partial_{\xi_j}. \]

b) Next, write the commutator for the electric field \(E\) in the following form

\[ [E_i(r, t), E_j(r', t')] = -\frac{1}{c^2} \frac{\partial^2}{\partial \tau^2} [A_i(r, t), A_j(r', t')]. \]

c) Finally, transform the commutator into a form which is proportional to \(\delta(\xi^2 - c^2 \tau^2)\).

Discuss the physical concept behind this solution.

Hint: It is not required to evaluate the derivatives \(\partial_{ij}\), it is sufficient to perform the integration over \(k\). This can be done by replacing the summation \(\frac{1}{V} \sum_k \to \int \frac{d^3k}{(2\pi)^3}\) by an integral.
Problem 2: Coherent states (Written)

Learning objective
In this problem you calculate some important properties of coherent states, which are quantum states that most closely resemble classical light. Coherent states are ubiquitous in quantum physics and are, for example, an important concept in quantum optics.

A coherent state $|\alpha\rangle$ (sometimes called Glauber state) is defined as the right-eigenstate of the (bosonic) annihilation operator $a$,

$$a |\alpha\rangle = \alpha |\alpha\rangle,$$

with eigenvalue $\alpha \in \mathbb{C}$.

a) Determine the coefficients $c_n(\alpha)$ of the expansion $|\alpha\rangle = \sum_{n=0}^{\infty} c_n(\alpha) |n\rangle$ for the normalized coherent state, $\langle \alpha | \alpha \rangle = 1$, where $|n\rangle$ is the eigenstate of the occupation operator $n = a^\dagger a$.

b) Introduce the displacement operator $\hat{D}(\alpha) = \exp(\alpha a^\dagger - \alpha^* a)$ and show that it creates a coherent state $|\alpha\rangle$ when applied to the vacuum $|0\rangle$.

Hint: Use the Baker-Campbell-Hausdorff (BCH) formula,

$$e^A e^B = e^{A + B + \frac{1}{2}[A,B]},$$

if the commutator $[A, B]$ commutes with $A$ and $B$.

c) Calculate the mean particle number $\langle n \rangle = \langle \alpha | n | \alpha \rangle$ and the variance $(\Delta n)^2 = \langle n^2 \rangle - \langle n \rangle^2$. To which probability distribution does this correspond?

d) Consider a one-dimensional harmonic oscillator with the Hamiltonian $H = \hbar \omega \left( n + \frac{1}{2} \right)$ in the initial state $|\psi(t = 0)\rangle = |\alpha_0\rangle$. Show that the time evolution can be written as $|\psi(t)\rangle = e^{i\phi(t)} |\alpha(t)\rangle$ with some time-dependent phase $\phi(t)$ and a time-dependent parameter $\alpha(t)$.

e) Compute the overlap $\langle \alpha | \alpha' \rangle$ and the operator $\int d^2 \alpha \ |\alpha\rangle \langle \alpha|$. Interpret your results.

Hint: The integral $\int d^2 \alpha = \int d(\text{Re} \alpha) \int d(\text{Im} \alpha)$ sums over all complex values of $\alpha$. Evaluate this integral in polar coordinates and use the Gamma function $\Gamma(x) = \int_0^\infty dz \ z^{x-1} e^{-z}$ with $\Gamma(n) = (n - 1)!$. 

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