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Problem 1: Commutator of the electric field — again ☺ (Oral)

> **Learning objective**
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> This problem deals with the quantized electric field. You calculate its commutator and show that it preserves causality.

In this problem, we calculate the commutator of the electric field, $[E_i(\mathbf{r},t)\,, E_j(\mathbf{r}',t')]$.

a) In a first step, calculate the commutator $[A_i(\mathbf{r},t)\,, A_j(\mathbf{r}',t')]$ of the vector potential.

Start by decomposing the vector potential $\mathbf{A}(\mathbf{r},t)$ into its normal modes

$$\mathbf{A}(\mathbf{r},t) = \sum_{\mathbf{k},\lambda} \left(\frac{hc^2}{V\omega_{\mathbf{k}}}\right)^{1/2} \left(a_{\mathbf{k},\lambda}\,\boldsymbol{\epsilon}(\mathbf{k},\lambda)\,e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}t)} + \text{h.c.}\right)\,, \tag{1}$$

where $\mathbf{r}$ is the spatial coordinate, t the time, $\mathbf{k}$ the wave vector, V the quantization volume, h Planck's constant, c the speed of light, and $\omega_{\mathbf{k}} = c|\mathbf{k}|$. $a_{\mathbf{k},\lambda}$ denotes the annihilation operator for wave number $\mathbf{k}$ and polarization λ, $\boldsymbol{\epsilon}$ is the vector of polarization.

In addition, use the completeness relation of the polarization vectors,

$$\sum_{\lambda} \epsilon_i(\mathbf{k},\lambda)\epsilon_j^*(\mathbf{k},\lambda) = \delta_{ij} - \frac{k_i k_j}{k^2}\,, \tag{2}$$

in order to bring the commutator into the following form:

$$[A_i(\mathbf{r},t), A_j(\mathbf{r}',t')] = \partial_{ij} K(\boldsymbol{\xi},\tau), \tag{3}$$

with $\boldsymbol{\xi} \equiv \mathbf{r} - \mathbf{r}'$ and $\tau \equiv t - t'$, where $K(\boldsymbol{\xi},\tau)$ has to be determined.

The differential operator ∂_{ij} is defined as

$$\partial_{ij} \equiv \frac{\partial_\tau^2}{c^2}\delta_{ij} - \partial_{\xi_i}\partial_{\xi_j}\,. \tag{4}$$

b) Next, write the commutator for the electric field $\mathbf{E}$ in the following form

$$[E_i(\mathbf{r},t), E_j(\mathbf{r}',t')] = -\frac{1}{c^2}\frac{\partial^2}{\partial\tau^2}[A_i(\mathbf{r},t), A_j(\mathbf{r}',t')]\,. \tag{5}$$

c) Finally, transform the commutator into a form which is proportional to $\delta\left(\xi^2 - c^2\,\tau^2\right)$.
Discuss the physical concept behind this solution.

Hint: It is not required to evaluate the derivatives ∂_{ij}, it is sufficient to perform the integration over $\mathbf{k}$. This can be done by replacing the summation $\frac{1}{V}\sum_{\mathbf{k}} \to \int \frac{\mathrm{d}^3k}{(2\pi)^3}$ by an integral.

Problem 2: Coherent states (Written)

> **Learning objective**
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> In this problem you calculate some important properties of *coherent states*, which are quantum states that most closely resemble classical light. Coherent states are ubiquitous in quantum physics and are, for example, an important concept in quantum optics.

A coherent state $|\alpha\rangle$ (sometimes called *Glauber state*) is defined as the right-eigenstate of the (bosonic) annihilation operator a,

$$a\,|\alpha\rangle = \alpha\;|\alpha\rangle\;, \tag{6}$$

with eigenvalue $\alpha \in \mathbb{C}$.

a) Determine the coefficients $c_n(\alpha)$ of the expansion $|\alpha\rangle = \sum_{n=0}^{\infty} c_n\,(\alpha)\;|n\rangle$ for the normalized coherent state, $\langle\alpha|\alpha\rangle = 1$, where $|n\rangle$ is the eigenstate of the occupation operator $n = a^\dagger a$.

b) Introduce the displacement operator $\hat{D}(\alpha) = \exp(\alpha\, a^\dagger - \alpha^*\, a)$ and show that it creates a coherent state $|\alpha\rangle$ when applied to the vacuum $|0\rangle$.

   Hint: Use the Baker-Campbell-Hausdorff (BCH) formula,

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]}\,, \tag{7}$$

   if the commutator $[A, B]$ commutes with A and B.

c) Calculate the mean particle number $\langle n\rangle = \langle\alpha|\,n\,|\alpha\rangle$ and the variance $(\Delta n)^2 = \langle n^2\rangle - \langle n\rangle^2$. To which probability distribution does this correspond?

d) Consider a one-dimensional harmonic oscillator with the Hamiltonian $H = \hbar\omega\left(n + \frac{1}{2}\right)$ in the initial state $|\psi(t = 0)\rangle = |\alpha_0\rangle$. Show that the time evolution can be written as $|\psi(t)\rangle = e^{i\phi(t)}\,|\alpha(t)\rangle$ with some time-dependent phase $\phi(t)$ and a time-dependent parameter $\alpha(t)$.

e) Compute the overlap $\langle\alpha|\alpha'\rangle$ and the operator $\int \mathrm{d}^2\alpha\;|\alpha\rangle\,\langle\alpha|$. Interprete your results.

   Hint: The integral $\int \mathrm{d}^2\alpha = \int \mathrm{d}(\mathrm{Re}\,\alpha) \int \mathrm{d}(\mathrm{Im}\,\alpha)$ sums over all *complex* values of α. Evaluate this integral in polar coordinates and use the Gamma function $\Gamma(x) = \int_0^\infty \mathrm{d}z\, z^{x-1} e^{-z}$ with $\Gamma(n) = (n-1)!$.