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Problem 1: The Casimir effect (Written, volunteer)

Owing to quantum fluctuations of the electromagnetic field, there is an attractive force between two parallel metallic plates separated by a distance d, even if the two plates are located in a vacuum and are electrically neutral. This is known as the *Casimir effect*. As we will see in this exercise, for two plates of area A separated by a distance d, the energy shift due to vacuum fluctuations is

$$U(d,A) = -\frac{\pi^2}{720}\frac{\hbar A}{d^3}. \tag{1}$$

Due to this energy shift, the force between the two plates is non zero and attractive

$$F = -\frac{\partial U(d,A)}{\partial d} = -\frac{\pi^2}{240}\frac{\hbar A}{d^4}. \tag{2}$$

This has been confirmed experimentally in 1958 by Sparnay (It was realized using $1\,\mathrm{cm}^2$ Chrome-Steal plates; at $d = 0.5\mu$ the attraction was $0.2\,\mathrm{dyn/cm}^2$).

a) Let us consider an electromagnetic field confined in a rectangular cavity (of dimensions $L_1 \times L_2 \times L_3$) with conducting walls. We must have $\mathbf{E}$ perpendicular and $\mathbf{B}$ tangential (the transverse component of the electric field vanishes at the surface of a perfect conductor). Show that these boundary conditions are satisfied by plane waves ($\sim e^{-i\omega t}$) if the components of the electric field have the following form

$$E_1 = E_1^0 \cos(k_1 x_1)\, \sin(k_2 x_2)\, \sin(k_3 x_3)\, e^{-i\omega t}, \tag{3}$$
$$E_2 = E_2^0 \sin(k_1 x_1)\, \cos(k_2 x_2)\, \sin(k_3 x_3)\, e^{-i\omega t}, \tag{4}$$
$$E_3 = E_3^0 \sin(k_1 x_1)\, \sin(k_2 x_2)\, \cos(k_3 x_3)\, e^{-i\omega t}, \tag{5}$$

where $k_i = n_i\pi/L_i$ and $n_i \in \mathbb{Z}$ and that the possible frequencies ω are restricted by the dispersion relation of light

$$\frac{1}{c^2}\omega^2(n_1,n_2,n_3) = \mathbf{k}^2 = \pi^2 \sum_i (n_i^2/L_i^2). \tag{6}$$

b) Show that the corresponding boundary conditions for the magnetic field $\mathbf{B}$ are fulfilled automatically. Recall that the magnetic field $\mathbf{B}$ is related to the electric field by the induction law $\nabla \times \mathbf{E} = i\,(\omega/c)\,\mathbf{B}$.

c) The amplitudes E_i^0 are fixed by the condition $\nabla \cdot \mathbf{E} = 0$, *i.e.* and thus satisfy

$$\sum_i E_i^0 k_i = 0. \tag{7}$$

Show that in general equation (7) has two linearly independent solutions, corresponding to the two polarizations of the electromagnetic field, except when one of the n_i vanish, that there is just one solution. If more than one vanish then there is no solution.

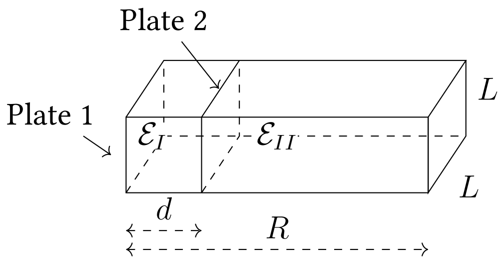

Abbildung 1: Setup of the two plates used to measure the Casimir effect.

d) Consider now two conducting and non-charged plates of dimensions $L \times L$ placed in a parallelogram with conducting walls as shown in the Figure 1. One conducting plate is fixed at the beginning of the box, while the second plate is chosen to be at a distance d from the former. This second plate will be moved to a distance R/η (with arbitrary $\eta > 0$) in a forthcoming step. We can define

$$U(d, L, R) \equiv \mathcal{E}_I(d) + \mathcal{E}_{II}(R-d) - [\mathcal{E}_{III}(R/\eta) + \mathcal{E}_{IV}(R - R/\eta)] \ , \tag{8}$$

as the energy difference between the zero point energies of the initial and final configurations, where $\mathcal{E}_I,\ \mathcal{E}_{II},\ \mathcal{E}_{III},\ \mathcal{E}_{IV}$ refer to the zero-point energy of each subspace, respectively. Show that each of them is divergent.

Defining these subspaces are indeed a tool to avoid divergences, as we are actually interested in taking the limit

$$U(d, L) = \lim_{R\to\infty} U(d, L, R) \ . \tag{9}$$

Thus, we need first to regularize the sums of the zero-point energy prior to calculation of Eq. (9). After the computation of Eq. (9), we will undo the regularization.

e) A convenient regularization method is the following

$$\mathcal{E}_{I,II} \to \mathcal{E}^{reg}_{I,II} = \sum_{\omega} \frac{1}{2}\hbar\omega \ \exp[-\alpha\omega/\pi c] \ . \tag{10}$$

Taking into account the dispersion relation (6) we have

$$\mathcal{E}^{reg}_I = \hbar c \sum_{l,m,n} k_{l,m,n}(d, L, L) \ \exp[-(\alpha/\pi)\, k_{l,m,n}(d, L, L))] \,, \tag{11}$$

where

$$k_{l,m,n}(d, L, L) = \sqrt{\left(\frac{l\pi}{d}\right)^2 + \left(\frac{m\pi}{L}\right)^2 + \left(\frac{n\pi}{L}\right)^2} \,. \tag{12}$$

We can define the regularized energy difference Eq. (8) as

$$U^{reg}(d, L, R, \alpha) = \mathcal{E}^{reg}_I(d) + \mathcal{E}^{reg}_{II}(R-d) - \{\mathcal{E}^{reg}_{III}(R/\eta) + \mathcal{E}^{reg}_{IV}(R - R/\eta)\} \ . \tag{13}$$

f) Let us consider the sum in equation (11). Consider very large L and replace the sums over m and n by integrals (a more precise way would be to study $U^{reg}(d,R,L^2,\alpha)/L^2$ when L goes to infinity) obtaining

$$\begin{aligned}\mathcal{E}_I^{reg}(d,L,\alpha) &= \hbar c \sum_{l=0}^{\infty} \int_0^{\infty} dm \int_0^{\infty} dn \sqrt{\left(\frac{l\pi}{d}\right)^2 + \left(\frac{m\pi}{L}\right)^2 + \left(\frac{n\pi}{L}\right)^2} \times \\ & \exp\left[-\frac{\alpha}{\pi}\sqrt{\left(\frac{l\pi}{d}\right)^2 + \left(\frac{m\pi}{L}\right)^2 + \left(\frac{n\pi}{L}\right)^2}\right] . \end{aligned} \tag{14}$$

In equation (13) the term with $l=0$ does not contribute to the sum. Therefore we can neglect it. Transform equation (14) into

$$\mathcal{E}_I^{reg} = -\frac{\pi^2}{4}\,\hbar c\, L^2\, \frac{d^3}{d\alpha^3} \sum_{l=1}^{\infty} \int_0^{\infty} \frac{dz}{1+z} \exp\left[-\frac{l}{d}\,\alpha\,\sqrt{1+z}\right] . \tag{15}$$

Perform the sum over l and then take the derivative with respect to α, arriving to

$$\mathcal{E}_I^{reg} = \frac{\pi^2 \hbar c L^2}{2d}\, \frac{d^2}{d\alpha^2}\, \frac{d/\alpha}{\exp[\alpha/d]-1} . \tag{16}$$

g) Calculate U^{reg} and obtain Eq. (1) by taking the limits

$$\lim_{R\to\infty} \lim_{\alpha\to 0} U^{reg}(d,L,R,\alpha) \tag{17}$$

Hint: $\frac{y}{e^y-1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} y^n$ where the B_n are the Bernoulli numbers.

Problem 2: Planck's radiation law (Oral)

Learning objective

In this problem, you derive Planck's radiation law of a black body which was a pioneering result in modern physics and quantum theory in particular.

First, consider a single mode of the electromagnetic field (without polarization) with Hamiltonian

$$H = \hbar\omega_k a_k^\dagger a_k . \tag{18}$$

a) Calculate the partition sum Z and write down the thermal state ρ at temperature T.

b) Calculate the mean particle number $\bar{n} = \langle n \rangle$ and the mean energy $\bar{E} = \langle H \rangle$ for the thermal state ρ.

In order to derive Planck's radiation law, consider a three-dimensional box of volume $V = L^3$ with periodic boundary conditions. The Hamiltonian of the system is now given by

$$H = \sum_{\mathbf{k},\lambda} \hbar\omega_k a_{\mathbf{k},\lambda}^\dagger a_{\mathbf{k},\lambda} , \tag{19}$$

where $\omega_k = c|\mathbf{k}|$ and λ is the polarization of the mode $\mathbf{k}$.

c) Based on your results in subtask b), calculate the spectral energy density $u_\omega(T)d\omega$ and the total energy density $u(T)$.

Hints:

- The system now consists of independent harmonic oscillators.
- The spectral energy density $u_\omega d\omega$ is given by the product of the energy and the density of states in the frequency interval $[\omega, \omega + d\omega]$.
- In order to calculate the density of states, first calculate the mode spacing and then take the limit $L \to \infty$.
- $\int_0^\infty dx \, \frac{x^3}{e^x - 1} = \frac{\pi^4}{15}$.