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Problem 1: Time-Dependent Perturbation Theory (Oral)

> **Learning objective**
>
> The first subtask is an application of time-dependent perturbation theory in first order. The goal is to identify the correct transition amplitude and evaluate the integrations analytically. The second subtask is a challenging calculation to derive the *exact* solution of the problem—a useful repetition/application of important methods in quantum mechanics. This problem is one of the few examples where the time evolution can be solved analytically (which allows for a verification of perturbation theory).

We investigate a one-dimensional harmonic oscillator with mass m, charge e, and frequency ω in a time-dependent electric field $E(t)$. The Hamiltonian is of the form

$$
\begin{aligned}
& H = H_0 + H'(t)\,, \\
\text{where} \quad & H_0 = \frac{p^2}{2m} + \frac{m}{2}\omega^2 x^2 \quad \text{(harmonic oscillator)} \\
\text{and} \quad & H'(t) = ex\,E(t) \quad \text{(perturbation)}\,.
\end{aligned}
\tag{1}
$$

The time-dependency of the external electric field is given by

$$
E(t) = \frac{A}{\tau\sqrt{\pi}}\, e^{-(t/\tau)^2}\,, \tag{2}
$$

where $A \in \mathbb{R}$ is a constant and $\tau > 0$ is the decay rate.

a) Calculate the transition probability $P_{0\to n}(t, t_0)$ from the ground state $|0\rangle$ at $t_0 = -\infty$ to an excited state $|n\rangle$ at $t = +\infty$ in first order perturbation theory. What happens for $\tau \to 0$?

   Hint: Use $x = \sqrt{\frac{\hbar}{2m\omega}}\,\left(a^\dagger + a\right)$ to evaluate the matrix element.

b) The transition probability can also be calculated *exactly*.
   Prove that the exact expression of $P_{0\to n}(t, t_0)$ for $t_0 = -\infty$ and $t = +\infty$ reads

$$
P_{0\to n} = \frac{K^{2n}}{n!} e^{-K^2} \quad \text{with} \quad K = \frac{eA}{\sqrt{2m\omega\hbar}}\, e^{-\omega^2\tau^2/4}\,, \tag{3}
$$

   and compare the result with a).

   Hint: Use the Dirac picture (as introduced in the lecture) and derive the equation of motion

$$
i\hbar\partial_t\,|\psi_D(t)\rangle = H'_D(t)\,|\psi_D(t)\rangle \quad \text{with} \quad H'_D(t) = \sqrt{\frac{\hbar}{2m\omega}}\,eE(t)\,\left[e^{-i\omega t}a + e^{i\omega t}a^\dagger\right]\,. \tag{4}
$$

   Then use the ansatz $|\psi_D(t)\rangle = \exp\left[-iK(t)a^\dagger\right]\left|\bar{\psi}(t)\right\rangle$ and choose $K(t)$ as

$$
K(t) = \frac{e}{\sqrt{2m\hbar\omega}} \int_{-\infty}^{t} e^{i\omega s}\, E(s)\,\mathrm{d}s \tag{5}
$$

to eliminate $a^\dagger$ from the Schrödinger equation. To this end, the relation $e^{iK(t)a^\dagger}\, a\, e^{-iK(t)a^\dagger} = a - iK(t)$ can be used to derive a differential equation for $\left|\bar{\psi}(t)\right\rangle$. Solve this equation with an ansatz $\left|\bar{\psi}(t)\right\rangle \propto |0\rangle$ to find the exact solution $|\psi(t)\rangle$ for the wave function in the Schrödinger picture. The transition probability is then $P_{0\to n} = |\,\langle n|\psi(t=\infty)\rangle\,|^2$.

Problem 2: Fermi's Golden Rule (Written)

Learning objective

This assignment is a repetition of the derivation of Fermi's golden rule. To this end, a system with a periodic perturbation in time is studied which requires only minor modifications of the proof already presented in the lecture.

Consider a system with Hamiltonian H_0, eigenstates $|n\rangle$ and eigenenergies E_n. We prepare the system in the initial state $|n = i\rangle$ at $t_0 = -\infty$ and slowly ramp up the periodic perturbation $V\cos(\omega t)$ with frequency ω and static perturbation V, i.e.,

$$H'(t) = V\cos(\omega t)\exp(\eta t) \tag{6}$$

for $\eta > 0$ (so that $H'(t) = 0$ for $t \to -\infty = t_0$).

Derive Fermi's golden rule in the adiabatic limit $\eta \to 0$, i.e., calculate the transition rate $\Gamma_{i\to f}$ from the initial state $|i\rangle$ to some final state $|f\rangle$ with energy E_f.

Hint: Use the Dirac sequence $\lim_{\eta\to 0} \frac{\eta}{x^2+\eta^2} = \pi\,\delta(x)$ and neglect the oscillating contributions to the transition rate (why is this reasonable?).