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Aufgabe 1: Liouville–von Neumann equation (Oral)

> **Learning objective**
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> In this problem, you will study the Liouville-von Neumann equation which describes the dynamics of the density matrix of a quantum system.

Let ρ be a density matrix given by $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$. Show that the time evolution of the density matrix is determined by the equation:

$$\partial_t \rho = -\frac{i}{\hbar}[H, \rho] \qquad \text{(Liouville-von Neumann equation)}. \tag{1}$$

Aufgabe 2: Partial trace (Oral)

> **Learning objective**
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> Here, you study the partial trace which is an operator-valued function used in quantum mechanics. It is used to obtain the reduced density matrix of a subsystem of a quantum system. It has very important applications in the context of open quantum systems, decoherence and quantum information.

Consider a quantum system S with Hilbert space $\mathcal{H}_S = \mathcal{H}_A \otimes \mathcal{H}_B$, which consists of two subsystems A and B. Let $\{|i\rangle_A\}$ and $\{|k\rangle_B\}$ be sets of orthonormal basis states in the respective Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$.

Note that each pure state $|\psi\rangle \in \mathcal{H}_S$ of the full system can be written as

$$|\psi\rangle = \sum_{i,k} c_{i,k} |i\rangle_A \otimes |k\rangle_B, \qquad \sum_{i,k} |c_{i,k}|^2 = 1. \tag{2}$$

Let $L(\mathcal{H}_i)$ be the space of linear operators over the Hilbert space $\mathcal{H}_i$. We define the *partial trace* (Partialspur) with respect to subsystem B:

$$\text{tr}_B[] : L(\mathcal{H}_S) \longrightarrow L(\mathcal{H}_A), \tag{3}$$

$$\text{tr}_B[X] = \sum_{m\in B} \langle m|_B X |m\rangle_B . \tag{4}$$

a) Let $M_A = M'_A \otimes \mathbb{1}_B \in L(\mathcal{H}_S)$ be an operator which acts only on subsystem A (and trivially on subsystem B). Show that the expectation value of M_A in the state $|\psi\rangle$ can be written as

$$\langle\psi|M_A|\psi\rangle = \text{tr}[M_A \cdot \rho] = \text{tr}[M'_A \cdot \rho_A] \tag{5}$$

where $\rho_A = \text{tr}_B[\rho] = \text{tr}_B[|\psi\rangle\langle\psi|]$ is the so-called *reduced density matrix*. Note that the right hand side is computed entirely in subsystem A.

b) As an example, consider a system which consists of two spin-1/2 subsystems. Let the full system be in the state

$$|\psi\rangle = u\,|\uparrow\rangle_A\,|\uparrow\rangle_B + v\,|\downarrow\rangle_A\,|\downarrow\rangle_B\,. \tag{6}$$

Determine the reduced density matrix ρ_A. Does ρ_A describe a pure state or a mixed state?

Aufgabe 3: Quantum computing: Fourier transform (Written, voluntary)

Learning objective

This problem deals with the quantum version of the Fourier transform used in many quantum algorithms as for example Shor's algorithm. Here, you will focus on the implementation of the quantum Fourier transform using so-called Hadamard gates.

a) Consider an N-dimensional Hilbert space with $N = 2^n$ states $|j\rangle$, where the index j runs from 0 to $2^n - 1$. We define the transformation U via

$$U\,|j\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \mathrm{e}^{2\pi i jk/N}\,|k\rangle\,. \tag{7}$$

Show that the operator U is unitary.

b) Let $|\psi\rangle = \sum_{j=0}^{N-1} x_j\,|j\rangle$ be a general state. Show that the amplitudes $y_k = \langle k|U|\psi\rangle$ are just the discrete Fourier transform of the amplitudes x_j.

c) We consider the case $n = 2$, i.e. $N = 4$. The four-dimensional Hilbert-space can be spanned by the product states $|s_1, s_2\rangle$, where the indices $s_i \in \{0, 1\}$ label the states of two qubits. The mapping between the basis $|s_1, s_2\rangle$ and the basis $|j\rangle$ is given by the binary representation of the number $j = 2s_1 + s_2$.

Show that the application of U can be written as

$$U\,|j\rangle = U\,|s_1, s_2\rangle = \frac{1}{2}\left(|0\rangle + \mathrm{e}^{2\pi i s_2/2}\,|1\rangle\right)\left(|0\rangle + \mathrm{e}^{2\pi i (s_1/2 + s_2/4)}\,|1\rangle\right). \tag{8}$$

Given this decomposition, we can see that U can be represented by single-qubit and two-qubit gates.

d) Start with the state $|j\rangle$ and apply the Hadamard gate (basis: $|0\rangle\,, |1\rangle$):

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{9}$$

to the first qubit. Show that this operation creates the state

$$\frac{1}{\sqrt{2}}\left(|0\rangle + \mathrm{e}^{2\pi i s_1/2}\,|1\rangle\right)|s_2\rangle\,. \tag{10}$$

e) We would now like to add a phase-shift of the form $\mathrm{e}^{2\pi i s_2/4}$, in order to create the state

$$\frac{1}{\sqrt{2}}\left(|0\rangle + \mathrm{e}^{2\pi i (s_1/2 + s_2/4)}\,|1\rangle\right)|s_2\rangle\,. \tag{11}$$

Which two-qubit gate creates this unitary transformation?

f) Finally, a Hadamard gate applied to the second qubit and a SWAP operation to exchange the two qubits can be used to complete the transformation U. How does U_{SWAP} look like?

g) Generalize this pattern to an arbitrary number n and show that the number of elementary operations (number of single- and two-qubit gates) scales like n^2. Compare this to the number of elementary arithmetic operations that a classical computer needs to compute the discrete Fourier transform.

   Note: There is no known way to efficiently measure the *amplitudes* y_k such that this would result in a speed-up over the classical Fourier transform. However, the quantum Fourier transform is used as a crucial step in many quantum algorithms.