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Problem 1: Stark Effect for a Harmonic Oscillator (Oral)

Learning objective

The goal of this problem is to apply a static perturbation (as derived in the lecture) up to second order for a simple setup. This setup has the special property that an exact solution exists and therefore the perturbation theory can be tested by a comparison with the exact solution.

The Hamiltonian of a one-dimensional harmonic oscillator within a homogeneous electric field E is given by

$$H = \frac{1}{2}\left(P^2 + Q^2\right) + eEQ\,, \tag{1}$$

where the Hamiltonian is dimensionless, i.e., $[Q, P] = i$. Consider the second term of the Hamiltonian as a perturbation of the free oscillator, that is, $H_1 = Q$ and $\lambda = eE$.

a) Calculate the perturbed eigenfunctions and energy eigenvalues up to second order in λ.
b) Compare the result of perturbation theory with the exact solution for the energy of the problem.

Problem 2: Fermi's Golden Rule (Written)

Learning objective

This assignment is a repetition of the derivation of Fermi's golden rule. To this end, a system with a periodic perturbation in time that is adiabatically switched on is studied; this requires some modifications of the proof presented in the lecture.

Consider a system with Hamiltonian H_0, eigenstates $|n\rangle$ and eigenenergies E_n. We prepare the system in the initial state $|n = i\rangle$ at $t_0 = -\infty$ and slowly ramp up the periodic perturbation $V\cos(\omega t)$ with frequency ω and static perturbation V, i.e.,

$$H'(t) = V\cos(\omega t)\exp(\eta t) \tag{2}$$

for $\eta > 0$ (so that $H'(t) = 0$ for $t \to -\infty = t_0$).

Derive Fermi's golden rule in the adiabatic limit $\eta \to 0$, i.e., calculate the transition rate $\Gamma_{i\to f}$ from the initial state $|i\rangle$ to some final state $|f\rangle$ with energy E_f.

Hint: Use the Dirac sequence $\lim_{\eta\to 0}\frac{\eta}{x^2+\eta^2} = \pi\,\delta(x)$ and neglect the oscillating contributions to the transition rate (why is this reasonable?).

Problem 3: Time-Dependent Perturbation Theory (Written)

Learning objective

The first subtask is an application of time-dependent perturbation theory in first order. The goal is to identify the correct transition amplitude and evaluate the integrations analytically. The second subtask is a challenging calculation to derive the *exact* solution of the problem—a useful repetition/application of important methods in quantum mechanics. This problem is one of the few examples where the time evolution can be solved analytically (which allows for a verification of perturbation theory).

We investigate a one-dimensional harmonic oscillator with mass m, charge e, and frequency ω in a time-dependent electric field $E(t)$. The Hamiltonian is of the form

$$\begin{aligned} & H = H_0 + H'(t)\,, \\ \text{where} \quad & H_0 = \frac{p^2}{2m} + \frac{m}{2}\omega^2 x^2 \quad \text{(harmonic oscillator)} \\ \text{and} \quad & H'(t) = ex\,E(t) \quad \text{(perturbation)}\,. \end{aligned} \tag{3}$$

The time-dependency of the external electric field is given by

$$E(t) = \frac{A}{\tau\sqrt{\pi}}\, e^{-(t/\tau)^2}\,, \tag{4}$$

where $A \in \mathbb{R}$ is a constant and $\tau > 0$ is the decay rate.

a) Calculate the transition probability $P_{0\to n}(t, t_0)$ from the ground state $|0\rangle$ at $t_0 = -\infty$ to an excited state $|n\rangle$ at $t = +\infty$ in first order perturbation theory. What happens for $\tau \to 0$?

   Hint: Use $x = \sqrt{\frac{\hbar}{2m\omega}}\,\left(a^\dagger + a\right)$ to evaluate the matrix element.

b) The transition probability can also be calculated *exactly*.
   Prove that the exact expression of $P_{0\to n}(t, t_0)$ for $t_0 = -\infty$ and $t = +\infty$ reads

$$P_{0\to n} = \frac{K^{2n}}{n!} e^{-K^2} \quad \text{with} \quad K = \frac{eA}{\sqrt{2m\omega\hbar}}\, e^{-\omega^2\tau^2/4}\,, \tag{5}$$

   and compare the result with a).

   Hint: Use the Dirac picture (as introduced in the lecture) and derive the equation of motion

$$i\hbar\partial_t \left|\psi_D(t)\right\rangle = H'_D(t)\left|\psi_D(t)\right\rangle \quad \text{with} \quad H'_D(t) = \sqrt{\frac{\hbar}{2m\omega}}\, eE(t)\, \left[e^{-i\omega t} a + e^{i\omega t} a^\dagger\right]\,. \tag{6}$$

   Then use the ansatz $\left|\psi_D(t)\right\rangle = \exp\left[-iK(t)a^\dagger\right]\left|\bar{\psi}(t)\right\rangle$ and choose $K(t)$ as

$$K(t) = \frac{e}{\sqrt{2m\hbar\omega}} \int_{-\infty}^{t} e^{i\omega s}\, E(s)\,\mathrm{d}s \tag{7}$$

   to eliminate $a^\dagger$ from the Schrödinger equation. To this end, the relation $e^{iK(t)a^\dagger}\, a\, e^{-iK(t)a^\dagger} = a - iK(t)$ can be used to derive a differential equation for $\left|\bar{\psi}(t)\right\rangle$. Solve this equation with an ansatz $\left|\bar{\psi}(t)\right\rangle \propto |0\rangle$ to find the exact solution $|\psi(t)\rangle$ for the wave function in the Schrödinger picture. The transition probability is then $P_{0\to n} = |\langle n|\psi(t=\infty)\rangle|^2$.