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Problem 1: Time-reversal symmetry (Oral)

Learning objective

According to Wigner's theorem, symmetry operators fall into two categories: unitary and anti-unitary operators. While symmetries are mostly described by unitary operators (e.g. $U(1)$ symmetry, rotational symmetry, translation symmetry), time-reversal symmetry is a fundamental (discrete) symmetry that is represented by a anti-unitary operator. In this problem, you will study the physics of time reversal for simple examples.

a) In the following, we consider a system with a time-independent Hamiltonian H that is invariant under time reversal given by the the operator T. Since T is connected to a symmetry of the system, it commutes with the Hamiltonian, $[H,T]=0$. The transformation of the time evolution operator $U(t)$ under time reversal is given by

$$T^{-1}U(t)T = U(-t)\,. \tag{1}$$

Show by using $U(t) = \exp(-\frac{i}{\hbar}Ht)$ that T is an anti-linear operator. Since T is anti-linear, Wigner's theorem implies that T is an anti-unitary operator.

Show further that if $|\psi\rangle$ is a solution of the Schrödinger equation, $T\,|\psi\rangle$ is a solution of the Schrödinger equation with $t \to -t$. Thus, $T\,|\psi\rangle$ satisfies the equation $-i\hbar\partial_t T\,|\psi\rangle = HT\,|\psi\rangle$.

Hint: An anti-linear operator has the property that $T(c\,|v\rangle) = c^* T\,|v\rangle$ for $c \in \mathbb{C}$ and $|v\rangle \in \mathcal{H}$ with some Hilbert space $\mathcal{H}$.

b) For spinless particles, the time-reversal operator T in the position basis satisfies

$$T\,|x\rangle = |x\rangle\ . \tag{2}$$

Show that $T\psi(x) = \psi^*(x)$. In order to do so, consider the action of T on some arbitrary state $|\psi\rangle$ and use $\psi(x) = \langle x|\psi\rangle$. It thus follows that in the position representation for spinless particles, $T = K$, where K denotes the complex conjugation with $Kc = c^*K$ for $c \in \mathbb{C}$. Show that consequently for spinless particles $T^2 = 1$.

c) Derive the transformation laws for the position, momentum and angular momentum operator in the position representation. How does this connect to classical physics? Show that the system is time-reversal invariant if the Hamiltonian is real, i.e. $H^* = H$.

d) Consider now a spin-1/2 particle. In this case, the time-reversal operator can be written as

$$T = \exp\left(-i\frac{\pi}{2}\sigma_y\right)K = -i\sigma_y\,K\,, \tag{3}$$

where σ_y is a Pauli matrix. Derive the transformation of the spin $\mathbf{S} = \frac{\hbar}{2}(\sigma_x, \sigma_y, \sigma_z)^T$ under the transformation (3) and show that $T^2 = -I$, where I is the identity operator.

e) **This task is optional and you can get bonus points solving it.**
Show that in a system that is time-reversal invariant and $T^2 = -I$ (as for example for a spin-1/2 particle), all energy levels are (at least) doubly degenerate. This is known as *Kramers' theorem.*

Problem 2: Properties of bosonic operators (Written)

Learning objective

The first part of this problem reviews properties of bosonic creation and annihilation operators in a single-mode Fock space, whereas the last part generalizes this concept to many-particle systems.

In order to solve the harmonic oscillator, one can introduce bosonic creation and annihilation operators satisfying the commutation relation

$$[b, b^\dagger] = 1 \,. \tag{4}$$

The occupation number operator is given by $\hat{n} = b^\dagger b$ with eigenstates $|n\rangle$ and eigenvalues n.

a) Using (4), show that $b\,|n\rangle$ and $b^\dagger\,|n\rangle$ are eigenstates of $\hat{n}$.

b) Prove the following relations:

$$b\,|n\rangle = \sqrt{n}\,|n-1\rangle \;, \tag{5}$$
$$b^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle \;. \tag{6}$$

c) Show that there has to be a state $|G\rangle$ with $b\,|G\rangle = 0$ and prove that n is an integer.
Hint: Use the fact that there are no states with negative norm.

d) Using the commutation relations of operators in bosonic Fock space,

$$[b_i, b_j^\dagger] = \delta_{ij}, \quad [b_i, b_j] = 0, \quad [b_i^\dagger, b_j^\dagger] = 0 \tag{7}$$

and the states

$$|n_1, \ldots, n_i, \ldots\rangle = \ldots \frac{(b_i^\dagger)^{n_i}}{\sqrt{n_i!}} \ldots \frac{(b_1)^\dagger}{\sqrt{n_1!}}\,|0\rangle \,, \tag{8}$$

prove the following relations:

$$b_i\,|n_1, \ldots, n_i, \ldots\rangle = \sqrt{n_i}\,|n_1, \ldots, n_i - 1, \ldots\rangle \;, \tag{9}$$
$$b_i^\dagger\,|n_1, \ldots, n_i, \ldots\rangle = \sqrt{n_i + 1}\,|n_1, \ldots, n_i + 1, \ldots\rangle \;. \tag{10}$$

e) Show that the states (8) build an orthonormal basis. Define the total number operator $\hat{n} = \sum \hat{n}_i = \sum b_i^\dagger b_i$. Now take a Hilbert-space of m-modes, i.e., the basis states of the Fock space are

$$|n_1, \ldots, n_m\rangle \,.$$

How many linearly independent states $|\Psi\rangle$ exist for a given particle number $N\,|\Psi\rangle = \hat{n}\,|\Psi\rangle$?

Problem 3: Properties of fermionic operators (Written)

Learning objective

This problem reviews properties of fermionic creation and annihilation operators in a single-mode Fock space, whereas the last part generalizes this concept to many-particle systems.

Now we introduce fermionic operators, which fulfill the anti-commutation relation

$$\{a, a^\dagger\} = 1\,. \tag{11}$$

The occupation number operator is given by $\hat{n} = a^\dagger a$ with eigenstates $|n\rangle$ and eigenvalues n.

a) Using (11), show that $a\,|n\rangle$ and $a^\dagger\,|n\rangle$ are eigenstates of $\hat{n}$.

b) Prove the following relations:

$$a\,|n\rangle = \sqrt{n}\,|1-n\rangle \qquad\qquad a^\dagger\,|n\rangle = \sqrt{1-n}\,|1-n\rangle\,. \tag{12}$$

c) Show that there has to be a state $|G\rangle$ with $a\,|G\rangle = 0$ and a state $|H\rangle$ with $a^\dagger\,|H\rangle = 0$. Further show that these are the only states in the Hilbert space. Assume that n is an integer.
Hint: Use the fact that there are no states with negative norm.

d) Using the anti-commutation relations of operators in fermionic Fock space,

$$\{a_i, a_j^\dagger\} = \delta_{ij}, \quad \{a_i, a_j\} = 0, \quad \{a_i^\dagger, a_j^\dagger\} = 0 \tag{13}$$

and the states

$$|n_1, \ldots, n_i, \ldots\rangle = \ldots (a_i^\dagger)^{n_i} \ldots (a_1)^\dagger\,|0\rangle\,, \tag{14}$$

prove the following relations:

$$a_i\,|n_1, \ldots, n_i, \ldots\rangle = \delta_{n_i,1}(-1)^{S_i}\,|n_1, \ldots, n_i - 1, \ldots\rangle\;, \tag{15}$$

$$a_i^\dagger\,|n_1, \ldots, n_i, \ldots\rangle = \delta_{n_i,0}(-1)^{S_i}\,|n_1, \ldots, n_i + 1, \ldots\rangle\;, \tag{16}$$

where $S_i = n_\infty + \cdots + n_{i+1}$

e) Show that the states (14) build an orthonormal basis. Define the total number operator $\hat{n} = \sum \hat{n}_i = \sum a_i^\dagger a_i$. Now take a Hilbert-space of m-modes, i.e., the basis states of the Fock space are

$$|n_1, \ldots, n_m\rangle\,.$$

How many linearly independent states $|\Psi\rangle$ exist for a given particle number $N\,|\Psi\rangle = \hat{n}\,|\Psi\rangle$?

Problem 4: Expectation values of bosonic and fermionic operators (Oral)

Learning objective

Here you will take the basis states of the bosonic and fermionic Fock space and practice the calculation of expectation values of many-particle systems.

In the following, we will work with the operators

$$\begin{aligned} A &= c_i^\dagger c_i & B &= c_i^\dagger c_i c_j^\dagger c_j \\ C &= c_i^\dagger c_j^\dagger c_j c_i & D &= c_i^\dagger c_j^\dagger c_i c_j . \end{aligned}$$

a) Show that these operators are self-adjoint. Consider the case $c_i = b_i$ (bosons) and $c_i = a_i$ (fermions) separately.

b) Calculate the expectation value of the operators A, B, C and D, taking the states (8), with $c_i = b_i$, and (14), with $c_i = a_i$.

c) Finally, determine the matrix element

$$\langle m_1, \ldots, m_i, \ldots | \, c_i^\dagger c_j + c_j^\dagger c_i \, | n_1, \ldots, n_i, \ldots \rangle \,,$$

again both for the bosonic and fermionic Fock space and operator algebra.