Prof. Dr. Hans Peter Büchler
Institute for Theoretical Physics III, University of Stuttgart

November 21st, 2019
WS 2019/20

Problem 1: Density matrix (Oral)

> **Learning objective**
>
> The purpose of this problem is to get familiar with the concept of the density matrix (operator) and calculate some important properties.

a) Show that $\rho^\dagger = \rho$ and $\mathrm{tr}(\rho) = 1$.

b) Show that $\mathrm{tr}(\rho^2) \leq 1$ and equality holds for a pure state. Show also that for a pure state $\rho^2 = \rho$.

c) Show that the expectation value of an operator O is given by $\langle O \rangle = \mathrm{tr}(\rho O)$.

d) Show that the equation of motion of the density matrix $\rho(t)$ of a system with Hamiltonian H is given by the von Neumann equation

$$i\hbar\partial_t \rho(t) = [H, \rho(t)] \, . \tag{1}$$

e) Consider a canonical ensemble at temperature T. Show that for a system with Hamiltonian H, the thermal density matrix is given by

$$\rho = \frac{1}{Z}\sum_n e^{-\beta E_n} \left|n\right\rangle \left\langle n\right| = \frac{1}{Z} e^{-\beta H} \, , \tag{2}$$

where $\beta = \frac{1}{k_B T}$, $Z = \mathrm{tr}(e^{-\beta H})$ is the partition function, and $H\left|n\right\rangle = E_n \left|n\right\rangle$.

Problem 2: Free fermions at finite temperature (Oral)

> **Learning objective**
>
> In this problem, you derive the Fermi-Dirac distribution. The distribution provides the average number of fermions in a single quantum state for a system of identical free fermions in thermodynamic equilibrium.

A system of free fermions is connected to a reservoir with which it can exchange energy and particles. Transferring a fermion from the system to the reservoir costs the energy μ (chemical potential). In the following, we consider a single quantum state $\left|\mathbf{p}\sigma\right\rangle$ of the system in thermodynamic equilibrium with momentum $\mathbf{p}$, spin σ, and energy $\epsilon_{\mathbf{p}}$. The state can be either occupied by one fermion or empty.

a) Show that the density matrix in the grand canonical ensemble at temperature T is given by

$$\rho_{\mathbf{p}\sigma} = \frac{1}{1 + e^{-\beta(\epsilon_{\mathbf{p}} - \mu)}} e^{-\beta(\epsilon_{\mathbf{p}}-\mu) c^\dagger_{\mathbf{p}\sigma} c_{\mathbf{p}\sigma}} \, , \tag{3}$$

with $\beta = \frac{1}{k_B T}$. Use this result to calculate $\left\langle n_{\mathbf{p}\sigma}\right\rangle$, which yields the Fermi-Dirac distribution. Plot the distribution as a function of $\epsilon_{\mathbf{p}}$ for different T.

b) Take the limit $T \to 0$ and interpret the result.

Problem 3: Coherent states (Written)

Learning objective

In this problem you calculate some important properties of *coherent states*, which are quantum states that most closely resemble classical light. Coherent states are ubiquitous in quantum physics and are, for example, an important concept in quantum optics.

A coherent state $|\alpha\rangle$ (sometimes called *Glauber state*) is defined as the right-eigenstate of the (bosonic) annihilation operator a,

$$a\,|\alpha\rangle = \alpha\;|\alpha\rangle\;, \tag{4}$$

with eigenvalue $\alpha \in \mathbb{C}$.

a) Determine the coefficients $c_n(\alpha)$ of the expansion $|\alpha\rangle = \sum_{n=0}^{\infty} c_n\,(\alpha)\;|n\rangle$ for the normalized coherent state, $\langle\alpha|\alpha\rangle = 1$, where $|n\rangle$ is the eigenstate of the occupation operator $n = a^\dagger a$.

b) Introduce the displacement operator $D(\alpha) = \exp(\alpha\, a^\dagger - \alpha^*\, a)$ and show that it creates a coherent state $|\alpha\rangle$ when applied to the vacuum $|0\rangle$.

   Hint: Use the Baker-Campbell-Hausdorff (BCH) formula,

   $$e^A e^B = e^{A+B+\frac{1}{2}[A,B]}\,, \tag{5}$$

   if the commutator $[A, B]$ commutes with A and B.

c) Calculate the mean particle number $\langle n\rangle = \langle\alpha|\, n\,|\alpha\rangle$ and the variance $(\Delta n)^2 = \langle n^2\rangle - \langle n\rangle^2$. To which probability distribution does this correspond?

d) Consider a one-dimensional harmonic oscillator with the Hamiltonian $H = \hbar\omega\left(n + \frac{1}{2}\right)$ in the initial state $|\psi(t=0)\rangle = |\alpha_0\rangle$. Show that the time evolution can be written as $|\psi(t)\rangle = e^{i\phi(t)}\,|\alpha(t)\rangle$ with some time-dependent phase $\phi(t)$ and a time-dependent parameter $\alpha(t)$.

e) Compute the overlap $\langle\alpha|\alpha'\rangle$ and the operator $\int \mathrm{d}^2\alpha\;|\alpha\rangle\,\langle\alpha|$. Interpret your results.

   Hint: The integral $\int \mathrm{d}^2\alpha = \int \mathrm{d}(\mathrm{Re}\,\alpha)\int \mathrm{d}(\mathrm{Im}\,\alpha)$ sums over all *complex* values of α. Evaluate this integral in polar coordinates and use the Gamma function $\Gamma(x) = \int_0^\infty \mathrm{d}z\, z^{x-1}e^{-z}$ with $\Gamma(n) = (n-1)!$.