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Problem 1: Spontaneous decay of the Hydrogen Atom (Oral)

Learning objective

In this exercise you will calculate which transitions in the Hydrogen atom can occur spontaneously. Basing on this you then will calculate the transition rates for electrons from the $n = 2$ manifold back to the ground state.

Consider an excited Hydrogen atom in the state $|n, l, m\rangle$ inside the electromagnetic vacuum. We try to find the possible transitions into a state $|n', l', m'\rangle$ by emitting a photon in the mode $|n_{k,\lambda} = 1\rangle$. Assuming the light-matter interaction is adiabatically turned on and off we can calculate the transition in first order perturbation theory as

$$\langle n', l', m'; n_{k,\lambda} = 1|\Psi(t)\rangle = \frac{1}{i\hbar} e^{-i\mathcal{E}_f(t-t_0)} \int_{t_0}^{t} dt_1 \, \langle n', l', m'; n_{k,\lambda} = 1| \, H_{\text{int}}(t_1) \, |n, l, m; n_{k,\lambda} = 0\rangle \, . \tag{1}$$

In the lecture you showed that this results in the transition rate

$$\begin{aligned} \Gamma &= \frac{d}{dt} |\langle n', l', m'; n_{k,\lambda} = 1|\Psi(t)\rangle|^2 \\ &= \frac{2\alpha\omega}{3} \frac{(\hbar\omega)^2}{mc^2 E_{\text{R}}} |r_{ab}/a_{\text{B}}|^2, \end{aligned}$$

where E_{R} is the Rydberg energy, a_{B} the Bohr radius and ω is the frequency resonant to the transition $E_n \to E_{n'}$. Further, the dipole matrix element $\mathbf{r}_{ab}$ is given by

$$\mathbf{r}_{ab} = \langle n', l', m'| \, \mathbf{r} \, |n, l, m\rangle \, . \tag{2}$$

a) First rewrite $\mathbf{r} = (x, y, z)^T = r(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)^T$. Then express this vector in terms of the spherical harmonics $Y_{l,m}(\theta, \phi)$. Use this to find the transitions which are allowed in first order.

b) Explicitly calculate the dipole matrix elements for the $n = 2 \to n = 1$ transition.

c) For an Hydrogen atom prepared in any of the $n = 2$ states, what are the average life times?

Problem 2: Symmetries of the Schrödinger equation (Written)

Learning objective

This problem deals with symmetries of the Schrödinger equation and how wave functions transform under associated symmetry transformations. In particular, you will show that the Schrödinger equation is invariant under Galilean transformations and gauge transformations (Eichtransformationen), the latter playing a crucial role in theoretical physics.

a) Consider two reference frames F and F' with coordinates (x, t) and (x', t'), respectively. The frame F' moves relatively to frame F with constant velocity v, such that

$$x = x' + vt', \quad t = t'. \tag{3}$$

The one-dimensional Schrödinger equation for a free particle of mass m in frame F is given by

$$i\hbar\partial_t\Psi(x,t) = -\frac{\hbar^2}{2m}\partial_x^2\Psi(x,t). \tag{4}$$

The Galilean transformation from $F \longrightarrow F'$ changes the wave function:

$$\Psi(x,t) \longrightarrow \Psi'(x',t') = \mathrm{e}^{if(x',t')}\,\Psi(x'+vt',t'), \tag{5}$$

where f is a real-valued function that depends on space and time. Determine the function f such that $\Psi'(x', t')$ fulfills the Schrödinger equation in frame F', that is show that the Schrödinger equation is invariant under Galilean transformations.

b) Let us now examine the behavior of the three-dimensional Schrödinger equation of a particle in an external electromagnetic field under a gauge transformation. For a massive particle of charge q, we have

$$\left[\frac{1}{2m}\left(\mathbf{p} - \frac{q}{c}\,\mathbf{A}\right)^2 + q\,\phi\right]\Psi = i\hbar\,\partial_t\Psi\,, \tag{6}$$

where $\mathbf{A}$ is the vector potential and ϕ is the scalar potential. We apply a gauge transformation

$$\mathbf{A} \longrightarrow \mathbf{A}' = \mathbf{A} + \nabla\chi\,, \tag{7}$$

$$\phi \longrightarrow \phi' = \phi - \frac{1}{c}\partial_t\chi, \tag{8}$$

where χ is a scalar-valued function. In order for the Schrödinger equation to be gauge invariant, we want

$$\left[\frac{1}{2m}\left(\mathbf{p} - \frac{q}{c}\,\mathbf{A}'\right)^2 + q\,\phi'\right]\Psi' = i\hbar\,\partial_t\Psi'. \tag{9}$$

How do you have to choose Ψ' in order for this equation to be consistent with equation (6)? Use the same approach as in the first part and discuss the similarities in both cases.

Problem 3: Photons as Fermions and violation of causality (Written and Bonus)

Learning objective

In this problem we show that the causality is violated if we assume a fermionic statistics for photons.

a) If photons would be Fermions, the creation and annihilation operators $a^\dagger_{\mathbf{k},\lambda}$, $a_{\mathbf{k},\lambda}$ anticommute and satisfy the anti-commutation relation $\{a_{\mathbf{k},\lambda}, a^\dagger_{\mathbf{k}',\lambda'}\} = \delta_{\mathbf{k},\mathbf{k}'}\delta_{\lambda,\lambda'}$. Calculate the anti commutator for the electric field **E**

$$\{E_i(\mathbf{r},t), E_j(\mathbf{r}',t')\} = -\frac{1}{c^2}\frac{\partial^2}{\partial\tau^2}\{A_i(\mathbf{r},t), A_j(\mathbf{r}',t')\}\,. \tag{10}$$

Focus on the equal time $\tau = 0$ and show that the anti commutator is non vanishing.
Hint: Follow the problem no. 2 from the previous exercise 07.

b) Causality requires that observables commute with each other for space like separation. Show that the intensity E^2 does not commute at equal time arising from the above results.
Hint: Calculate the commutator $\left[E_i^2(\mathbf{r}), E_j^2(\mathbf{r}')\right]$ and make use of anti-commutation relation for $E_i(\mathbf{r})$ and $E_j(\mathbf{r}')$.