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Problem 1: The Casimir effect (Written and Bonus)

Learning objective

In this problem, you study the Casimir effect which is a quantum effect that, owing to quantum fluctuations of the electromagnetic field, leads to an attractive force between two parallel conducting plates in vacuum. This effect was predicted by H. Casimir in 1948[a] and experimentally confirmed by Sparnaay in 1958[b].

[a] Hendrik Casimir, *On the attraction between two perfectly conducting plates.* Proc. Kon. Nederland. Akad. Wetensch. B51, 793 (1948)
[b] M.J. Sparnaay, *Measurements of attractive forces between flat plates*, Physica 24, 751-764 (1958)

a) Let us consider an electromagnetic field confined in a rectangular cavity (of dimensions $L_1 \times L_2 \times L_3$) with conducting walls. Since the transverse component of the electric field vanishes at the surface of a perfect conductor, **E** has to be perpendicular and **B** parallel to the walls at the boundaries. Show that these boundary conditions are satisfied by plane waves ($\sim e^{-i\omega t}$) if the components of the electric field have the following form

$$E_1 = E_1^0 \cos(k_1 x_1)\, \sin(k_2 x_2)\, \sin(k_3 x_3)\, e^{-i\omega t}\,, \tag{1}$$
$$E_2 = E_2^0 \sin(k_1 x_1)\, \cos(k_2 x_2)\, \sin(k_3 x_3)\, e^{-i\omega t}\,, \tag{2}$$
$$E_3 = E_3^0 \sin(k_1 x_1)\, \sin(k_2 x_2)\, \cos(k_3 x_3)\, e^{-i\omega t}\,, \tag{3}$$

where $k_i = n_i \pi / L_i$ and $n_i \in \mathbb{Z}$ and that the possible frequencies ω are restricted by the dispersion relation of light

$$\frac{1}{c^2}\omega^2(n_1, n_2, n_3) = \mathbf{k}^2 = \pi^2 \sum_i (n_i^2 / L_i^2). \tag{4}$$

b) Show that the corresponding boundary conditions for the magnetic field **B** are fulfilled automatically. Recall that the magnetic field **B** is related to the electric field by the Maxwell-Faraday equation $\nabla \times \mathbf{E} = i\,(\omega/c)\,\mathbf{B}$.

c) The amplitudes E_i^0 are fixed by the condition $\nabla \cdot \mathbf{E} = 0$ and thus satisfy

$$\sum_i E_i^0 k_i = 0. \tag{5}$$

Show that in general equation (5) has two linearly independent solutions, corresponding to the two polarizations of the electromagnetic field. Show that if one of the n_i vanishes, there is only one solution while there is no solution if two or more vanish.

d) Consider now two conducting and non-charged, conducting plates of dimensions $L \times L$ placed parallel to each other as shown in Figure 1. One conducting plate is fixed at the beginning of the

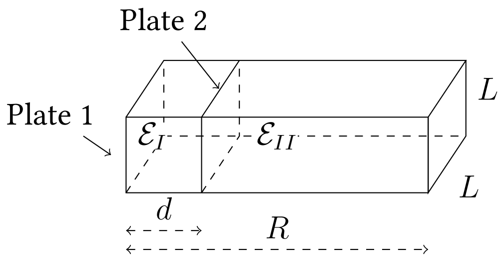

Figure 1: Setup of the two plates used to measure the Casimir effect.

box, while the second plate is chosen to be at a distance d from the former. This second plate will be moved to a distance R/η (with arbitrary $\eta > 0$) in a forthcoming step. We can define

$$U(d, L, R) := \mathcal{E}_I(d) + \mathcal{E}_{II}(R-d) - [\mathcal{E}_{III}(R/\eta) + \mathcal{E}_{IV}(R - R/\eta)] \ , \tag{6}$$

as the energy difference between the zero point energies of the initial and final configurations, where $\mathcal{E}_I,\ \mathcal{E}_{II},\ \mathcal{E}_{III},\ \mathcal{E}_{IV}$ refer to the zero-point energy of each section, respectively. Show that each of the energies is divergent.

Defining these sections is indeed a tool to avoid divergences, as we are actually interested in taking the limit

$$U(d, L) = \lim_{R\to\infty} U(d, L, R) \ . \tag{7}$$

Therefore, we first regularize the sums of the zero-point energy before calculating Eq. (7). In the end, we undo the regularization in order to arrive at the final result.

As the divergence comes from contributions from high-frequencies, a convenient regularization method is to introduce some high-frequency cut-off

$$\mathcal{E}_{I,II} \to \mathcal{E}_{I,II}^{reg} = \sum_{\omega} \frac{1}{2}\,\hbar\omega\ \exp[-\alpha\omega/\pi c] \ , \tag{8}$$

where the limit $\alpha \to 0$ corresponds to the case we are interested in. Taking into account the dispersion relation (4), we have

$$\mathcal{E}_I^{reg} = \hbar c \sum_{l,m,n} k_{l,m,n}\ \exp[-\,(\alpha/\pi)\,k_{l,m,n})] \, , \tag{9}$$

where

$$k_{l,m,n} = \sqrt{\left(\frac{l\pi}{d}\right)^2 + \left(\frac{m\pi}{L}\right)^2 + \left(\frac{n\pi}{L}\right)^2} \, . \tag{10}$$

The regularized energy difference is then defined as

$$U^{reg}\,(d, L, R, \alpha) = \mathcal{E}_I^{reg}(d) + \mathcal{E}_{II}^{reg}(R-d) - [\mathcal{E}_{III}^{reg}(R/\eta) + \mathcal{E}_{IV}^{reg}(R - R/\eta)] \ . \tag{11}$$

Consider now the sum in equation (9). For large L, one can replace the sums over m and n by integrals (a more precise way would be to study $U^{reg}(d, R, L^2, \alpha)/L^2$ when L goes to infinity) obtaining

$$\begin{aligned}\mathcal{E}_I^{reg}(d, L, \alpha) &= \hbar c \sum_{l=0}^{\infty} \int_0^{\infty} \mathrm{d}m \int_0^{\infty} \mathrm{d}n \sqrt{\left(\frac{l\pi}{d}\right)^2 + \left(\frac{m\pi}{L}\right)^2 + \left(\frac{n\pi}{L}\right)^2} \times \\ &\quad \exp\left[-\frac{\alpha}{\pi}\sqrt{\left(\frac{l\pi}{d}\right)^2 + \left(\frac{m\pi}{L}\right)^2 + \left(\frac{n\pi}{L}\right)^2}\right] . \end{aligned} \tag{12}$$

As the term with $l = 0$ in (12) only depends on α, it does not contribute to the energy difference (11) and can simply be neglected in the following.

e) Show that equation (12) can be written as

$$\mathcal{E}_I^{reg} = -\frac{\pi^2}{4}\,\hbar c\, L^2\, \frac{\mathrm{d}^3}{\mathrm{d}\alpha^3} \sum_{l=1}^{\infty} \int_0^{\infty} \frac{\mathrm{d}z}{1+z} \exp\left[-\frac{l}{d}\,\alpha\,\sqrt{1+z}\right] . \tag{13}$$

Perform the summation over l and then take the derivative with respect to α, arriving to

$$\mathcal{E}_I^{reg} = \frac{\pi^2 \hbar c L^2}{2d} \frac{\mathrm{d}^2}{\mathrm{d}\alpha^2} \frac{d/\alpha}{\exp[\alpha/d] - 1} . \tag{14}$$

f) Calculate U^{reg} by taking the limits

$$\lim_{R\to\infty} \lim_{\alpha \to 0} U^{reg}(d, L, R, \alpha) \tag{15}$$

and show that the energy shift due to the vacuum fluctuations is given by

$$U(d, A) = -\frac{\pi^2}{720} \frac{\hbar c A}{d^3}, \tag{16}$$

where $A = L^2$ is the surface of one plate. Due to this energy shift, there is a non-zero, attractive force between the two plates

$$F = -\frac{\partial U(d, A)}{\partial d} = -\frac{\pi^2}{240} \frac{\hbar c A}{d^4} . \tag{17}$$

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Hint: $\frac{y}{e^y - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} y^n$ where the B_n are the Bernoulli numbers.