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January 30th, 2020
WS 2019/20

Learning objective

On this problem set, you will study two decoherence effects: decay and dephasing. Decoherence effects usually appear when a smaller system, for example consisting of only a few particles, is coupled to some large bath as for example the continuum of a quantized radiation field. If the system and the bath are initially uncorrelated and the correlation times of the bath are much smaller than the typical timescales of the system, the time evolution of the system alone is described by a *Lindblad* master equation. As an example, you will consider a two-level system that is subject to both spontaneous decay and dephasing and you will learn the differences between them.

Problem 1: Lindblad master equation I: Decay of a two-level system (Oral)

Consider a two-level system with a ground state $|g\rangle$ and an excited state $|e\rangle$. This system is equivalent to a spin-$\frac{1}{2}$ system and we define the spin operators

$$\sigma_x = |e\rangle\langle g| + |g\rangle\langle e| \ , \qquad \sigma_y = -i\,|e\rangle\langle g| + i\,|g\rangle\langle e| \ , \qquad \sigma_z = |e\rangle\langle e| - |g\rangle\langle g| \ . \tag{1}$$

They satisfy the commutation and anti-commutation relations

$$[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k \qquad \text{and} \qquad \{\sigma_i, \sigma_j\} = 2\delta_{ij} \ , \tag{2}$$

respectively. Further, we define the transition operators

$$\sigma^+ = |e\rangle\langle g| = \frac{1}{2}(\sigma_x + i\sigma_y) \qquad \text{and} \qquad \sigma^- = |g\rangle\langle e| = \frac{1}{2}(\sigma_x - i\sigma_y) \, . \tag{3}$$

The two-level system is now coupled to a bath and the description of the dynamics of the two-level system in terms of a Lindblad master equation reads

$$\partial_t \rho(t) = \Gamma\left(\sigma^- \rho(t) \sigma^+ - \frac{1}{2}\left\{\sigma^+\sigma^-, \rho(t)\right\}\right) \ , \tag{4}$$

where Γ denotes the spontaneous emission rate of the excited state.

a) Show that the density matrix of a two-level system can be represented as

$$\rho(t) = \frac{1}{2}\left(I + \mathbf{v}(t)\cdot\boldsymbol{\sigma}\right) \ , \tag{5}$$

where I is the 2×2 identity matrix and $\mathbf{v}(t) = \langle\boldsymbol{\sigma}(t)\rangle = \mathrm{tr}(\boldsymbol{\sigma}\rho(t))$ denotes the Bloch vector. The matrix elements $\rho_{11}(t)$ and $\rho_{22}(t)$ are the populations of the excited and ground state, respectively. The off-diagonal elements $\rho_{12}(t) = \rho_{21}^*(t)$ are the coherences between ground and excited state.

b) Derive the equations of motion for the components of the Bloch vector and calculate the stationary solution for the population of the excited state.

c) Calculate the time-dependent solution for the the coherences and the populations of the ground and excited state for the case when the system initially is (i) in the ground state and (ii) in the excited state. Sketch the trajectory of the Bloch vector on the Bloch sphere in both cases. Show also that any mixed state eventually ends up in a pure state.

Problem 2: Lindblad master equation II: Dephasing in a two-level system (Oral)

In this problem, we study the dephasing of the two-level system considered in the previous problem. The Lindblad master equation describing the dephasing is given by

$$\partial_t \rho(t) = \frac{\gamma}{2}\left(\sigma_z \rho(t) \sigma_z - \rho(t)\right) = \frac{\gamma}{2}\left(\sigma_z \rho(t) \sigma_z - \frac{1}{2}\left\{\sigma_z \sigma_z, \rho(t)\right\}\right), \tag{6}$$

where γ is the dephasing rate.

a) Derive the equations of motion for the components of the Bloch vector $\mathbf{v}(t)$. How does the population of the excited state evolve over time? How does this compare to the result obtained in the previous problem?

b) Consider the state $|\psi\rangle = (|g\rangle + |e\rangle)/\sqrt{2}$ and calculate its time evolution in the long-time limit, that is $\gamma t \gg 1$. Is this a pure or a mixed state?

c) Now, include the spontaneous emission from Problem 1 and write down the equations of motion for the Bloch vector $\mathbf{v}(t)$. Calculate the analytical solution and discuss your result.