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You can find detailed information about the lecture and the tutorials on the website `http://www.itp3.uni-stuttgart.de/teaching/qm/index.html`. There are two types of problems: *Written* problems have to be handed in via ILIAS and will be corrected by the tutors. *Oral* problems have to be prepared for the exercise session and will be presented by the tutors during the online tutorials. You might be asked questions to them. To be admitted to the exam, we require 80% of the written problems to be solved or treated adequately and 66% of the oral problems to be prepared.

Problem 1: Plane waves (Written, 2 points)

Learning objective

This problem serves as a repetition of basic properties of plane waves. They will prove useful for many calculations in quantum mechanics.

In an interval $[0, L]$ with periodic boundary conditions, plane waves are given by

$$\psi_n = \frac{1}{\sqrt{L}} \exp\left(\frac{i}{\hbar} p_n x\right), \quad \text{where} \quad p_n = \frac{2\pi\hbar n}{L} \quad \text{and} \quad n \in \mathbb{Z}. \tag{1}$$

a) Show that they form a complete set of orthonormal basis functions. In other words, show that:

$$\int_0^L dx\, \psi_n^*(x)\psi_m(x) = \delta_{n,m} \qquad \text{(orthonormality)}\,, \tag{2}$$

$$\sum_{n=-\infty}^{\infty} \psi_n^*(x)\psi_n(x') = \delta(x - x') \qquad \text{(completeness)}\,. \tag{3}$$

b) In the limit $L \to \infty$, the orthogonality and completeness relations read

$$\int_{-\infty}^{\infty} dx\, \exp\left[-\frac{i}{\hbar}(p - p')x\right] = 2\pi\hbar\, \delta(p - p') \quad \text{(orthogonality)}\,, \tag{4}$$

$$\int_{-\infty}^{\infty} \frac{dp}{2\pi\hbar}\, \exp\left[\frac{i}{\hbar}(x - x')p\right] = \delta(x - x') \quad \text{(completeness)}\,. \tag{5}$$

Perform the limit $L \to \infty$ explicitly for ψ_n by using an appropriate prefactor and show both the orthogonality (4) and the completeness (5). (Note that the basis functions are no longer normalizable and are thus called a *general set of basis functions*.)

Problem 2: Fourier transform (Written, 6 points)

Learning objective

In this problem, which serves as a repetition, you will prove important properties of the Fourier transform.

Given a complex-valued function $f : \mathbb{R} \to \mathbb{C}$, the one-dimensional Fourier transform is defined as

$$\mathcal{F}[f(x)] = \int_{-\infty}^{\infty} \mathrm{d}x \, e^{-ikx} f(x) \equiv \hat{f}(k) \ . \tag{6}$$

a) Show that

$$\mathcal{F}[1] = 2\pi\delta(k) \quad \text{and} \quad \mathcal{F}[\delta(x)] = 1 \ , \tag{7}$$

where $\delta(q)$ is the Dirac delta function. Use $\int_{-\infty}^{\infty} \mathrm{d}q \, h(q)\delta(q) = h(0)$.

b) Show that the inverse Fourier transform is

$$\mathcal{F}^{-1}[\hat{f}(k)] = \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} \, e^{ikx} \hat{f}(k) = f(x) \ . \tag{8}$$

c) Show that

$$\mathcal{F}[f(x+a)] = e^{ika} \, \mathcal{F}[f(x)] \ . \tag{9}$$

d) Show that

$$\mathcal{F}[\partial_x f(x)] = ik \, \mathcal{F}[f(x)] \ . \tag{10}$$

e) The convolution of two functions $f : \mathbb{R} \to \mathbb{C}$ and $g : \mathbb{R} \to \mathbb{C}$ is defined as

$$(f * g)(x) = \int_{-\infty}^{\infty} \mathrm{d}\xi \, f(\xi) g(x-\xi) = \int_{-\infty}^{\infty} \mathrm{d}\xi \, f(x-\xi) g(\xi) \ . \tag{11}$$

Show that

$$\mathcal{F}[(f * g)(x)] = \mathcal{F}[f(x)] \cdot \mathcal{F}[g(x)] \ . \tag{12}$$

f) Calculate the Fourier transform of the Gaussian function

$$f(x) = e^{-ax^2} \quad \text{with} \quad a \neq 0 \ , \ \mathrm{Re}(a) \geq 0 \ , \ a \in \mathbb{C} \ . \tag{13}$$

Pay particular attention to the calculation for a purely imaginary a.

Problem 3: Classical action (Oral)

Learning objective

This problem deals with the classical action and Lagrangian which serves both as a repetition of classical mechanics as well as a starting point for deriving dynamical equations in quantum mechanics.

In classical mechanics, the action S is defined as

$$S[q](t) = \int_{t_0}^{t} dt'\, L(\dot{q}(t'), q(t'), t')\,, \tag{14}$$

where L is the *Lagrangian function* (also called *Lagrangian*) and q is some generalized coordinate of the system.

a) Calculate the action $S(q_\tau, \tau)$ along classical paths for the following systems:
   - free particle,
   - harmonic oscillator with $V(q) = \frac{1}{2}m\omega^2 q^2$,
   - constant force F,

   where $q(t = \tau_0) = 0$ is the starting point and $q(t = \tau) = q_\tau$ is the endpoint of the path at some time $\tau > \tau_0$.

b) For classical paths with a fixed starting point $q(\tau_0) = q_0$, one can interpret S as a function of q_τ and τ, i.e. $S = S(q_\tau, \tau)$. Show that

$$\frac{\partial S}{\partial q_\tau} = p_\tau\,, \quad \frac{\partial S}{\partial \tau} = -H\,, \tag{15}$$

   where H is the Hamiltonian function of the system.

   Hint: For some fixed starting point q_0, each path can be parametrized by some arbitrary point q_τ on that path such that $q(t)$ itself can be thought of as a function of both q_τ and some general time t.