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Problem 1: Commutators (Written, 3 points)

> **Learning objective**
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> The commutator of operators plays a crucial role in quantum mechanics. The goal of this exercise is to learn and train to evaluate the commutator of a few basic operators.

The commutator of linear operators $A, B \in L(\mathcal{H})$ acting on some Hilbert space $\mathcal{H}$ is defined as

$$[A, B] := AB - BA. \tag{1}$$

a) Let A, B, and C be linear operators on $\mathcal{H}$. Show that

$$[AB, C] = A[B, C] + [A, C]B \tag{2a}$$
$$[A, BC] = [A, B]C + B[A, C]. \tag{2b}$$

b) Let x and $p = -i\hbar\partial_x$ be the position and momentum operator, as introduced in the lecture. Evaluate the commutators

$$[x, p], \quad [x, p^2], \quad [x^2, p^2], \quad [xp, p^2]. \tag{3}$$

c) Now let f and g be smooth functions with a Taylor series representation on $\mathbb{R}$. Show that

$$[p, g(x)] = -i\hbar\frac{dg(x)}{dx} \quad \text{and} \quad [x, f(p)] = i\hbar\frac{df(p)}{dp} \tag{4}$$

where $f(A) \equiv \sum_{n=0}^{\infty} f_n A^n$ with an arbitrary operator A and Taylor coefficients f_n.

Problem 2: Baker-Campbell-Hausdorff formula (Written, 2 points)

> **Learning objective**
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> The Baker-Campell-Hausdorff formula is an important relation, which is used many times in quantum mechanics in different context. In this exercise, this important formula is derived

Consider two non-commuting operators A and B, for which the relations $[A, [A, B]] = [B, [A, B]] = 0$ are satisfied.

a) Show that for the operators A and B, the equality

$$e^{-At}Be^{At} = B - t[A, B] \tag{5}$$

is satisfied, where t is an arbitrary number.

b) Show that both operators satisfy the Baker-Campbell-Hausdorff formula

$$e^{A+B} = e^A e^B e^{-\frac{[A,B]}{2}}. \tag{6}$$

Hint: Derive a first-order differential equation for the operator $W(t) = e^{-tA} e^{t(A+B)}$ and solve it.

Problem 3: Heisenberg uncertainty relations (Oral, 2 points)

> **Learning objective**
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> In this exercise, we will derive the Heisenberg uncertainty relation for the position and momentum operator. We will then, generalize this relation for two Hermitian operators.

a) Show that the momentum p and position x operators satisfy the uncertainty principle

$$\Delta p \cdot \Delta x \geq \frac{\hbar}{2} \tag{7}$$

where we have defined the *standard deviation* $\Delta A = \sqrt{\langle (A - \langle A \rangle)^2 \rangle}$ of an operator A of expectation value $\langle A \rangle$.

Hint: Look up or derive your own the proof of Heisenberg uncertainty principle which is different from the proof done in lecture.

b) Given two Hermitian operators A and B, prove the generalized uncertainty principle

$$(\Delta A)^2 \cdot (\Delta B)^2 \geq \frac{|\langle \mathrm{i}\,[A, B] \rangle|^2}{4}. \tag{8}$$

Show that this result is consistent with task a).