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Problem 1: Decay of Metastable States (Written, 4 points + 3 bonus points ($\star$))

> **Learning objective**
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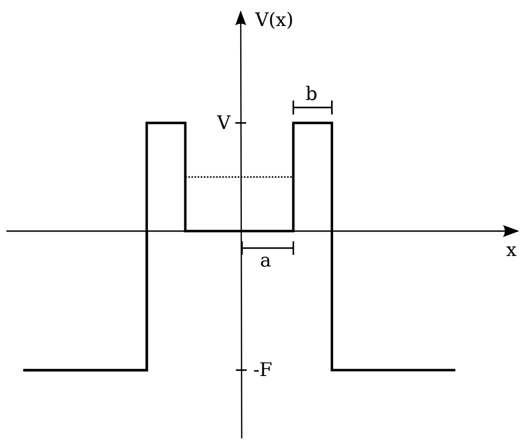
> Tunneling is an important quantum mechanical phenomenon with wide macroscopic implications, one example is radioactive decay. Double well potentials are often employed to model the decay of metastable states caused by tunneling. In this exercise you will study this phenomena employing a simple symmetric potential well with finite walls and depth (double well in one direction). You will find that expanding equation (2) determines the energy eigenvalues, and the decay rate can be related to their imaginary part.

We consider a particle in the potential $V(x)$ (see figure below). It can be expected that for $F \gg V$ and $V \gg E > 0$ the system possesses states that correspond to bound states, but which are not stable and can decay through quantum mechanical tunneling out of the central potential well. In the following we restrict ourselves to symmetric wave functions, which are described by the ansatz:

$$\begin{aligned} |x| < a: \quad & \psi(x) = \cos(qx) \\ a < |x| < a+b: \quad & \psi(x) = A\exp\left[-\kappa(|x|-a)\right] + B\exp\left[\kappa(|x|-a)\right] \\ |x| > a+b: \quad & \psi(x) = C\exp\left[ik(|x|-a-b)\right] \end{aligned} \tag{1}$$

with $q = \sqrt{2mE}/\hbar$, $\kappa = \sqrt{2m(V-E)}/\hbar$ and $k = \sqrt{2m(E+F)}/\hbar$. Note that this ansatz contains only an *outgoing* plane wave for $|x| > a+b$. Such an ansatz introduces boundary conditions which violate the hermiticity of the Hamiltonian, i.e. a finite probability current is leaving the system and the norm of the wave function is no longer conserved. As a consequence the eigenenergies have an imaginary part. These wave functions are termed *metastable states*.

a) Formulate the continuity conditions for the wave function $\psi(x)$ and its derivative $\psi'(x)$ at each potential step and show that the following implicit equation determines the eigenenergies E_n.



Expand in the small parameter κ/k.

$$q\sin(qa) = \kappa(A-B) = \kappa\cos(qa)\left[\coth(\kappa b) + \frac{\kappa}{ik\sinh(\kappa b)^2} + \mathcal{O}\left((\kappa/k)^2\right)\right] \tag{2}$$

We consider a large barrier, i.e. tunneling is exponentially suppressed by $\exp(-2\kappa b)$. Therefore expand Eq. (2) in the small parameter $\exp(-2\kappa b)$.

b) To zeroth order in $\exp(-2\kappa b)$ the eigenenergies are those of a potential well of finite depth. Show that for $q/\kappa \ll 1$ the lowest eigenergy E_0 has the following form

$$E_0 = \frac{\hbar^2 q_0^2}{2m} \quad \text{with} \quad q_0 = \frac{\pi/2}{a+1/\kappa}. \tag{3}$$

c) To first order in $\exp(-2\kappa b)$ the energy E_{ms} can be written as

$$E_{\text{ms}} = E_0 + \Delta - i\Gamma/2. \tag{4}$$

Determine Δ and Γ. Show that the imaginary part of the energy can be interpreted as a decay rate

$$\langle\psi(t)|\psi(t)\rangle \sim \exp(-\Gamma t). \tag{5}$$

Note that $|\psi(t)\rangle$ is the wavefunction inside the well.

d) Show that the probability current density is given by the following relations

$$j(x=a+b, t=0) = \frac{\hbar k}{m}|\psi(a+b,0)|^2/N = \frac{\Gamma}{2\hbar}. \tag{6}$$

What is a meaningful normalization N of the wavefunction ?

e)* Now we consider the true eigenenergies of the potential $V(x)$ which respect the hermiticity of the Hamiltonian. Such solutions are characterized by an ingoing and outgoing wave for $|x| > a+b$. The ground state and first excited state in a symmetric potential behave asymptotically (for $|x| \to \infty$) like

$$\psi_0 \sim \cos(|x|k + \delta_0)$$
$$\psi_1 \sim \text{sgn}(x) i \sin(|x|k + \delta_1)$$

where

$$\text{sgn}(x) = \begin{cases} +1 & \text{if } x > 1 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 1. \end{cases}$$

The phases δ_0 and δ_1 are the *scattering phases* of the symmetric and antisymmetric wave functions, respectively.

Write down the ansatz for the symmetric wavefunction for the potential $V(x)$ and formulate the continuity conditions, which determine the scattering phase $\delta_0(E)$ for the energy E. Compare the equations with the expressions from task a).

f)* We define the scattering cross section $\sigma = \sigma_0 + \sigma_1$, where the partial scattering cross sections σ_i describe scattering with the corresponding symmetry of the wavefunction. The optical theorem expresses the partial scattering cross sections in terms of the corresponding scattering phases

$$\sigma_i = \frac{1}{(\tan \delta_i)^2 + 1}. \tag{7}$$

Prove that the partial scattering cross section exhibits poles at the complex energies E_{ms} and $E^\star_{\text{ms}}$.

g)* Show that in the vicinity of the poles $\sigma_0(E)$ takes the following form

$$\sigma_0(E) \sim \frac{1}{(E - E_0 + \Delta)^2 + \Gamma^2/4}. \tag{8}$$

This shows that metastable states result in resonances in the partial scattering cross sections.

Problem 2: Creation and annihilation operators (Written, 3 points)

Learning objective

The harmonic oscillator problem is a corner stone in physics. In a previous problem set you saw how we can solve the 1-dimensional harmonic oscillator using the path integral formalism. Here we introduce another approach to solve the harmonic oscillator problem using ladder operators. You can realize that calculations become easier. Instead of solving differential equations, we have a linear algebra problem where we need to find the eigensystem of our Hamiltonian.

Given the 2-dimensional harmonic oscillator Hamiltonian:

$$H = \hbar\omega_+(a_+^\dagger a_+ + \frac{1}{2}) + \hbar\omega_-(a_-^\dagger a_- + \frac{1}{2}); \tag{9}$$

where the creation and annihilation operators $a_\pm^\dagger$, $a_\pm$ satisfy the following commutation rules:

$$\left[a_\pm, a_\pm^\dagger\right] = 1; \qquad [a_\pm, a_\pm] = \left[a_\pm^\dagger, a_\pm^\dagger\right] = 0; \tag{10}$$

$$\left[a_\pm, a_\mp^\dagger\right] = [a_\pm, a_\mp] = \left[a_\pm^\dagger, a_\mp^\dagger\right] = 0; \tag{11}$$

a) Show that the Hamiltonian of the system is diagonal with respect to the eigenstates of the number operators $N_+ = a_+^\dagger a_+$ and $N_- = a_-^\dagger a_-$ and find the respective eigenenergies. Does the measurement of the observable $N = N_+ + N_-$ specify the state of the system?

b) Define the groundstate of the system $|0,0\rangle$ through $a_\pm |0,0\rangle = 0$, correctly normalized $\langle 0,0|0,0\rangle = 1$. The Hilbert space can be constructed by the application of creation operators on $|0,0\rangle$. Show that the commutation relations

$$[N_+, a_+] = -a_+ \qquad \left[N_+, a_+^\dagger\right] = a_+^\dagger \tag{12}$$

hold (same for N_-). Find the normalized eigenvectors $|n_+, n_-\rangle$ of both number operators with eigenvalues n_+ and n_-.

c) Verify the following relations:

$$a_+^\dagger |n_+, n_-\rangle = \sqrt{n_+ + 1}\, |n_+ + 1, n_-\rangle \tag{13}$$

$$a_+ |n_+, n_-\rangle = \sqrt{n_+}\, |n_+ - 1, n_-\rangle \tag{14}$$

Problem 3: Coherent states (Oral, 5 points)

Learning objective

The eigenstates of the harmonic oscillator Hamiltonian $|n\rangle$, are not eigenstates of the ladder operators. The coherent state which is an eigenstate of the annihilation operator is a useful object for example in quantum optics. In this exercise you investigate some properties of coherent states and realize their convenience especially when describing the dynamic behavior of a quantum harmonic oscillator.

For every $\alpha \in \mathbb{C}$, we define the coherent state

$$|\psi_\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n \geq 0} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \tag{15}$$

being $|n\rangle$ an eigenstate of the 1-dimensional harmonic oscillator Hamiltonian.

a) Verify that the coherent state $|\psi_\alpha\rangle$ is an eigenstate of the annihilation operator where α is the eigenvalue, then show that the creation operator has no eigenstates.

b) Show that

$$|\psi_\alpha\rangle = e^{\alpha a^\dagger - \alpha^* a} |0\rangle \tag{16}$$

where $a^\dagger$ and a are creation and annihilation operators, respectively.
(Tip: $e^{-\alpha a} |0\rangle = |0\rangle$)

c) Calculate $\langle x \rangle_\alpha = \langle \psi_\alpha | x | \psi_\alpha \rangle$, $\langle p \rangle_\alpha$, Δx_α, Δp_α and show that, for all $\alpha \in \mathbb{C}$,

$$\Delta x_\alpha \Delta p_\alpha = \frac{\hbar}{2} \tag{17}$$

is valid, *i.e.*, coherent states minimize the position and momentum uncertanty relation. (Tip: $a |\psi_\alpha\rangle = \alpha |\psi_\alpha\rangle$)

d) Show that coherent states are not orthogonal and the relation

$$\langle \psi_\alpha | \psi_\beta \rangle = e^{-\frac{1}{2}(|\alpha|^2 + |\beta|^2 - 2\alpha^* \beta)} \tag{18}$$

is valid.

e) Derive the time evolution of coherent state $|\psi_\alpha(t)\rangle$ and of the expectation values $\langle x \rangle_t$ and $\langle p \rangle_t$ under the Hamiltonian $H = \hbar\omega \left(a^\dagger a + \frac{1}{2}\right)$.