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Problem 1: Linear algebra basics (Written, 6 points)

Learning objective

In this exercise we study the properties of Hermitian and unitary operators, which play a fundamental role in the description of quantum mechanics.

Let $\mathcal{H}$ be a finite-dimensional Hilbert space whose elements are denoted by $|\psi\rangle$, $|\phi\rangle$, ... Further, we consider linear operators A, B on that Hilbert space.

a) The *Hermitian conjugate* (or *adjoint*) of A (also denoted $A^\dagger$) is defined as

$$\langle\psi|A^\dagger\phi\rangle = \langle A\psi|\phi\rangle = \langle\phi|A\psi\rangle^* . \tag{1}$$

Show that $(AB)^\dagger = B^\dagger A^\dagger$.

b) An operator satisfying $A = A^\dagger$ is called *Hermitian* or *self-adjoint*. Show that the eigenvalues of Hermitian operators are real and the corresponding eigenvectors of two different eigenvalues are orthogonal. (*Remark*: You can assume that there are no degenerate eigenvalues.)

c) A (linear) operator that satisfies $UU^\dagger = U^\dagger U = I$ with I the identity, or equivalently $U^{-1} = U^\dagger$, is called a *unitary* operator. Show that unitary operators leave the norm of a vector unchanged. Show that the eigenvalues λ_n of a unitary operator have modulus unity, i.e. $\lambda_n = e^{i\phi_n}$ with $\phi_n \in \mathbb{R}$, and that eigenvectors corresponding to different eigenvalues are orthogonal.

d) Let A be a Hermitian operator. Show that the operator $U = e^{i\alpha A}$, $\alpha \in \mathbb{R}$, is unitary.

e) Consider an orthonormal basis set $\{|n\rangle\}$ and another basis set $\{|n'\rangle\}$ with $|n'\rangle = U\,|n\rangle$, where U is a unitary operator. Show that $\{|n'\rangle\}$ is also orthonormal. If we denote the matrix elements of an operator A by $A_{mn} = \langle m|A|n\rangle$, how do the matrix elements A_{mn} relate to the matrix elements in the basis $\{|n'\rangle\}$?

f) Let A and B be Hermitian operators with $[A, B] = 0$. Show that A and B can be diagonalized simultaneously. (*Remark:* Neglect the case of degenerate eigenvalues.)

Problem 2: Position space and momentum space representation (Oral)

Learning objective

The difficulty of many problems in physics can be drastically reduced by choosing the appropriate basis. The representation of operators depends on that choice. In this exercise we study the position space and momentum space basis in detail.

Consider the position space basis $\{|x\rangle\}$ which is an eigenbasis of the position operator $\hat{x}$ with eigenvalues x and the momentum space basis $\{|p\rangle\}$ which is an eigenbasis of the momentum operator $\hat{p}$ with eigenvalues p. The spectra of both operators are continuous such that the normalization condition and the completeness relation read

$$\langle\lambda'|\lambda\rangle = \delta(\lambda' - \lambda)\,, \qquad I = \int d\lambda\; |\lambda\rangle\langle\lambda| \qquad \lambda = x, p\,. \tag{2}$$

a) Using the canonical commutation relation, show that $e^{\frac{i}{\hbar}\hat{p}a}\hat{x}e^{-\frac{i}{\hbar}\hat{p}a} = \hat{x} + aI$. Show that $e^{-\frac{i}{\hbar}\hat{p}a}|x\rangle$ is an eigenstate of $\hat{x}$ with eigenvalue $x + a$, i.e. $e^{-\frac{i}{\hbar}\hat{p}a}|x\rangle = |x + a\rangle$.

b) Let $\langle x|\phi\rangle = \phi(x)$ be the component of a physical state vector $|\phi\rangle$ in the basis $|x\rangle$. Calculate the matrix elements of the operators $\hat{x}$ and $e^{-\frac{i}{\hbar}\hat{p}a}$ in the position space basis and deduce the representation of the momentum operator in position space.

c) Determine the wave functions $\psi_p(x)$ of the eigenvectors $|p\rangle$ in the position space representation. (*Hint:* Choose the normalization factor such that (2) is fulfilled.)

d) Let $\langle p|\phi\rangle = \tilde{\phi}(p)$ be the component of a physical state vector $|\phi\rangle$ in the basis $|p\rangle$. Find the action of the operators $\hat{x}$ and $\hat{p}$ in the momentum space representation.

e) Show that $e^{\frac{i}{\hbar}\hat{p}a}f(\hat{x})e^{-\frac{i}{\hbar}\hat{p}a} = f(\hat{x} + aI)$ and $e^{-a\partial_x}f(x) = f(x - a)$.

Problem 3: Schrödinger equation in position and momentum space (Written, 3 points)

Learning objective

In this exercise we combine the results of the previous exercises. First we show that the time evolution is given by a unitary operator and then consider the Schrödinger equation in position and momentum space.

a) Show that the time evolution of a state $|\psi\rangle$ satisfying the Schrödinger equation with a time-independent Hamiltonian $\hat{H}$ is given by the time evolution operator $\hat{U}(t) = e^{-\frac{i}{\hbar}\hat{H}t}$.

b) Consider a Hamiltonian of the form $\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})$. Write down the Schrödinger equation in position and momentum space.

c) Consider a potential of the form $V(x) = g\delta(x)$ $(g < 0)$ in position space. Calculate the eigenvalue and the wave function of the bound state by solving the time-independent Schrödinger equation in momentum representation. (*Hint:* You may assume that the wave function decays fast enough in momentum space as well).

Problem 4: Angular momentum (Oral)

Learning objective

After the intensive study of the position and momentum operator, we focus on the angular momentum operator in the following exercises. With its help we can describe rotations and introduce the concepts of scalar and vector operators.

Consider the angular momentum operator $\mathbf{L} = \mathbf{r} \times \mathbf{p}$. In the lecture it was shown that $\mathbf{L}$ is the infinitesimal generator of rotations such that rotations around some axis $\mathbf{n}$ with $\mathbf{n}^2 = 1$ about some angle ω can be written as $U_\omega = \exp\left(-i\omega \mathbf{L} \cdot \mathbf{n}/\hbar\right)$. If U_ω is the operator performing a rotation around some axis $\omega = \omega\mathbf{n}$ in the Hilbert space, i.e. $|\phi_\omega\rangle = U_\omega |\phi\rangle$; a *scalar* operator S transforms like

$$U_\omega^\dagger \, S \, U_\omega = S \,, \tag{3}$$

and a *vector* operator $\mathbf{X}$ transforms like

$$U_\omega^\dagger \, \mathbf{X} \, U_\omega = R_\omega \, \mathbf{X} \,, \tag{4}$$

where R_ω is the usual rotation matrix in three dimensions around some axis ω.

a) Show that for a scalar operator S, $[\mathbf{L}, S] = 0$.

b) Using that $\mathbf{r}$ and $\mathbf{p}$ are vector operators, show that $\mathbf{L}$ is also a vector operator. (*Hint:* Consider the components of $U_\omega^\dagger \, \mathbf{r} \times \mathbf{p} \, U_\omega$ and show that $U_\omega^\dagger \, \mathbf{r} \times \mathbf{p} \, U_\omega = U_\omega^\dagger \, \mathbf{r} \, U_\omega \times U_\omega^\dagger \, \mathbf{p} \, U_\omega$.)

c) Show that $[\mathbf{L}, \mathbf{p} \cdot \mathbf{r}] = 0$ on the one hand by explicitely calculating the commutator and on the other hand by showing that $\mathbf{p} \cdot \mathbf{r}$ is a scalar operator.