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Problem 1: The Born Series (Oral, 3 points)

Learning objective

In this exercise, we study the Born series and derive the general expression for the expansions in real and Fourier space. Each term in the expansion can be very graphically expressed with strict rules how to related the diagram with the mathematical expression. This is an example of the famous Feynman diagrams.

a) Take the Ansatz

$$\psi_{\mathbf{k}}(\mathbf{r}) = \sum_{n=0}^{\infty} \delta\psi_{\mathbf{k}}^{(n)}(\mathbf{r}), \qquad \delta\psi_{\mathbf{k}}^{(0)}(\mathbf{r}) = \psi_0(\mathbf{r}) = e^{i\mathbf{kr}} \tag{1}$$

for the solution of the Lippmann-Schwinger equation. Assuming the Ansatz converges, derive (e.g. by induction) the equivalent representation of the solution

$$\psi_{\mathbf{k}}(\mathbf{r}) = \psi_0(\mathbf{r}) + \sum_{n=1}^{\infty} X^n \psi_0(\mathbf{r}). \tag{2}$$

What is the operator X and its action on a wave function $X\psi$?

b) Find X in momentum space and compare it to the real space representation.

c) Scattering processes are typically depicted as Feynman diagrams. Below you can see such a diagram. Explain the meaning of each part of the diagram and give a way to translate between diagrams and the scattering series (2), the so called Feynman rules.

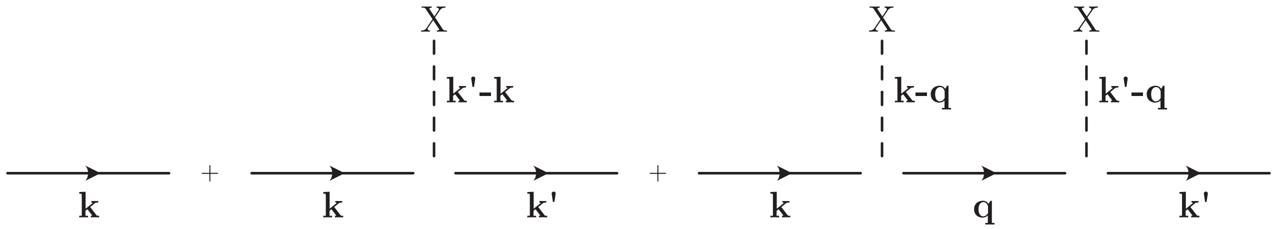

Figure 1: Feynman diagram for the Born approximation with the potential $V(\mathbf{r})$.

Problem 2: Scattering at the Yukawa-Potential (Written, 3 points)

Learning objective

The goal of this exercise is to determine the scattering amplitude within the Born approximation for the Yukawa Potential. The Yukawa potential describes the interaction between charged particles in a theory, where photons have a mass, and the Coulomb potential is recovered for vanishing photon mass $\mu \to 0$. Note, that for the scattering amplitude something strange happens in this limit as the Coulomb potential is a long-range interaction.

a) The Yukawa-potential is given by

$$V(r) = g\frac{\exp(-\mu r)}{r}. \tag{3}$$

In the first order Born approximation, calculate the scattering amplitude.

b) For a generic potential $V(\mathbf{r})$ derive the general expression for the second order contribution to the scattering amplitude and express it by the Fourier components of V.

c) Determine the scattering amplitude in the second order Born approximation for the Yukawa potential.

Hint: Bring the scattering amplitude to the form

$$f^{(2)}_{\mathbf{k}}(\mathbf{k}') \propto \int d^3q \frac{1}{(\mathbf{q}-\mathbf{k})^2+\mu^2}\frac{1}{k^2-q^2+i\delta}\frac{1}{(\mathbf{k}'-\mathbf{q})^2+\mu^2}.$$

Then you can use the relation

$$\int_{-1}^{1} dz \frac{2}{[(a+b)+z(a-b)]^2} = \frac{1}{ab}.$$

with $a = (\mathbf{q}-\mathbf{k})^2+\mu^2$ and $b = (\mathbf{q}-\mathbf{k}')^2+\mu^2$ to simplify the $\mathbf{q}$ integration.

Problem 3: Exact scattering at a Pseudo Potential (Written, 3 points)

Learning objective

The low energy scattering for all short range interactions is determined by a single parameter: the s-wave scattering length. Here, we study an effective potential, which reproduces exactly this low energy scattering amplitude, and which can be used in describing the scattering behavior of particles at low temperatures.

The potential

$$V(\mathbf{r}) = g\delta(\mathbf{r})\partial_r r \tag{4}$$

is commonly used to model low energy scattering. Since it contains a derivative it is refered to as a pseudo-potential.

a) First calculate $\int d^3r V(r)\psi(\mathbf{r})$. Consider two examples: a smooth wavefunction and a wavefunction with a $1/r$ divergence at $\mathbf{r} = 0$, like a spherical wave for example. Explain the role of the $\partial_r r$ term in the potential.

b) Use the Ansatz

$$\psi_{\mathbf{k}}(\mathbf{r}) = e^{i\mathbf{k}\mathbf{r}} + f(\Omega)e^{ikr}/r \tag{5}$$

to solve the Lippmann-Schwinger equation for the pseudo-potential $V(\mathbf{r})$.

c) The scattering length a is defined as

$$\lim_{k\to 0} \sigma = 4\pi a^2. \tag{6}$$

Given a potential $U(\mathbf{r})$ with scattering length a, find the coupling strength g such that the pseudo potential $V(\mathbf{r})$ accurately describes the low energy scattering at the potential U.