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Problem 1: Stark Effect in the Harmonic Oscillator (Oral, 2 points)

Learning objective

In this exercise, we investigate how a static homogeneous electric field affects the eigenfunctions of the one-dimensional harmonic oscillator. Using perturbation theory, we calculate the perturbed eigenfunctions and eigenenergies and compare them to the exact solution of the problem.

The Hamiltonian operator of a one-dimensional harmonic oscillator in a homogeneous electric field E reads

$$H = \frac{1}{2}(P^2 + Q^2) + eEQ. \tag{1}$$

Consider the second term in the Hamiltonian as a perturbation of the free harmonic oscillator with $H_1 = Q$ and $\lambda = eE$.

a) Compute the perturbed eigenfunctions in first order and the perturbed eigenenergies up to second order in λ.

b) Using the substitution $Y = Q + \lambda$, find the exact eigenvalues of the Hamiltonian and compare your result with (a).

Problem 2: Perturbation in 2-level System (Oral, 3 points)

Learning objective

Similar to the previous exercise, we calculate eigenfunctions and eigenenergies in perturbation theory and compare the result to the exact solution. In this exercise, however, we are also interested in the degenerate case, where two eigenenergies of the unperturbed system coincide.

The unperturbed Hamiltonian operator for a 2-level system in its eigenbasis reads

$$H_0 = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}. \tag{2}$$

Consider the perturbed Hamiltonian operator of the form

$$H(\lambda) = H_0 + \lambda\, \mathbf{e} \cdot \boldsymbol{\sigma} \tag{3}$$

with the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{4}$$

and $\mathbf{e} = (e_x, e_y, e_z)$ a general vector.

a) In the non-degenerate case ($E_1 \neq E_2$), compute the eigenfunctions in first order and the eigenenergies up to second order in λ.

b) In the degenerate case ($E_1 = E_2$), find a linear combination of unperturbed eigenvectors which diagonalizes the perturbation $\mathbf{e} \cdot \boldsymbol{\sigma}$. Calculate the first order correction to the eigenenergies.

c) Solve the Schrödinger equation

$$H(\lambda)\psi(\lambda) = E(\lambda)\psi(\lambda) \tag{5}$$

exactly and compare your result with (a) and (b).

Problem 3: Projection Operator (Oral, 3 points)

Learning objective

With the projection operator we can project a sate onto a subspace of the Hilbert space, which has many applications in quantum mechanics, e.g. it can be used to construct an appropriate basis for a given symmetry. In this exercise, we define the projection operator and show its basic properties.

Define the projection operator P as $P^2 = P$ with $P^\dagger = P$.

a) Show that the eigenvalues are 0 and 1.

b) Given the subspace generated by $|\psi_i\rangle$ ($i = 1, ..., N$). Show that $P = \sum_i |\psi_i\rangle \langle\psi_i|$ is a projection operator for the subspace.

c) Given an initial state $|\psi\rangle$, consider the measurement of P (as given in (b)) with observed value 1. What is the state after the measurement?