a) Let us consider an electromagnetic field confined in a rectangular cavity (of dimensions $L_1 \times L_2 \times L_3$) with conducting walls. Since the transverse component of the electric field vanishes at the surface of a perfect conductor, $E$ has to be perpendicular and $B$ parallel to the walls at the boundaries. Show that these boundary conditions are satisfied by plane waves ($\sim e^{-i\omega t}$) if the components of the electric field have the following form

\[ E_1 = E_1^0 \cos(k_1x_1) \sin(k_2x_2) \sin(k_3x_3) e^{-i\omega t}, \]
\[ E_2 = E_2^0 \sin(k_1x_1) \cos(k_2x_2) \sin(k_3x_3) e^{-i\omega t}, \]
\[ E_3 = E_3^0 \sin(k_1x_1) \sin(k_2x_2) \cos(k_3x_3) e^{-i\omega t}, \]  

where $k_i = n_i \pi / L_i$ and $n_i \in \mathbb{Z}$ and that the possible frequencies $\omega$ are restricted by the dispersion relation of light

\[ \frac{1}{c^2} \omega^2 (n_1, n_2, n_3) = k^2 = \pi^2 \sum_i (n_i^2 / L_i^2). \]  

b) Show that the corresponding boundary conditions for the magnetic field $B$ are fulfilled automatically. Recall that the magnetic field $B$ is related to the electric field by the Maxwell-Faraday equation $\nabla \times E = i(\omega/c) B$.

c) The amplitudes $E_i^0$ are fixed by the condition $\nabla \cdot E = 0$ and thus satisfy

\[ \sum_i E_i^0 k_i = 0. \]  

Show that in general equation (5) has two linearly independent solutions, corresponding to the two polarizations of the electromagnetic field. Show that if one of the $n_i$ vanishes, there is only one solution while there is no solution if two or more vanish.

d) Consider now two conducting and non-charged, conducting plates of dimensions $L \times L$ placed parallel to each other as shown in Figure 1. One conducting plate is fixed at the beginning of the
Figure 1: Setup of the two plates used to measure the Casimir effect.

box, while the second plate is chosen to be at a distance $d$ from the former. This second plate will be moved to a distance $R/\eta$ (with arbitrary $\eta > 0$) in a forthcoming step. We can define

$$U(d, L, R) := \mathcal{E}_I(d) + \mathcal{E}_{II}(R - d) - \left[\mathcal{E}_{III}(R/\eta) + \mathcal{E}_{IV}(R - R/\eta)\right],$$  \hspace{1cm} (6)

as the energy difference between the zero point energies of the initial and final configurations, where $\mathcal{E}_I$, $\mathcal{E}_{II}$, $\mathcal{E}_{III}$, $\mathcal{E}_{IV}$ refer to the zero-point energy of each section, respectively. Show that each of the energies is divergent.

Defining these sections is indeed a tool to avoid divergences, as we are actually interested in taking the limit

$$U(d, L) = \lim_{R \to \infty} U(d, L, R).$$  \hspace{1cm} (7)

Therefore, we first regularize the sums of the zero-point energy before calculating Eq. (7). In the end, we undo the regularization in order to arrive at the final result.

As the divergence comes from contributions from high-frequencies, a convenient regularization method is to introduce some high-frequency cut-off

$$\mathcal{E}_{I,II} \to \mathcal{E}_{I,II}^{\text{reg}} = \sum_{\omega} \frac{1}{2} \hbar \omega \exp\left[-\alpha \omega / \pi c\right],$$  \hspace{1cm} (8)

where the limit $\alpha \to 0$ corresponds to the case we are interested in. Taking into account the dispersion relation (4), we have

$$\mathcal{E}_I^{\text{reg}} = \hbar c \sum_{l,m,n} k_{l,m,n} \exp[-(\alpha / \pi) k_{l,m,n}],$$  \hspace{1cm} (9)

where

$$k_{l,m,n} = \sqrt{\left(\frac{l \pi}{d}\right)^2 + \left(\frac{m \pi}{L}\right)^2 + \left(\frac{n \pi}{L}\right)^2}.$$  \hspace{1cm} (10)

The regularized energy difference is then defined as

$$U^{\text{reg}}(d, L, R, \alpha) = \mathcal{E}_I^{\text{reg}}(d) + \mathcal{E}_{II}^{\text{reg}}(R - d) - \left[\mathcal{E}_{III}^{\text{reg}}(R/\eta) + \mathcal{E}_{IV}^{\text{reg}}(R - R/\eta)\right].$$  \hspace{1cm} (11)
Consider now the sum in equation (9). For large $L$, one can replace the sums over $m$ and $n$ by integrals (a more precise way would be to study $\frac{U_{reg}(d, R, L^2, \alpha)}{L^2}$ when $L$ goes to infinity) obtaining

$$E_{reg}^I(d, L, \alpha) = \frac{\hbar c}{4} \sum_{l=0}^{\infty} \int_0^\infty \, dm \int_0^\infty \, dn \sqrt{\left(\frac{l\pi}{d}\right)^2 + \left(\frac{m\pi}{L}\right)^2 + \left(\frac{n\pi}{L}\right)^2} \times$$

$$\exp \left[ -\frac{\alpha}{\pi} \sqrt{\left(\frac{l\pi}{d}\right)^2 + \left(\frac{m\pi}{L}\right)^2 + \left(\frac{n\pi}{L}\right)^2} \right].$$

(12)

As the term with $l = 0$ in (12) only depends on $\alpha$, it does not contribute to the energy difference (11) and can simply be neglected in the following.

e) Show that equation (12) can be written as

$$E_{reg}^I = -\frac{\pi^2}{4} \frac{\hbar c L^2}{d^3} \frac{d^3}{d\alpha^3} \sum_{l=1}^{\infty} \int_0^\infty \frac{dz}{1+z} \exp \left[ -\frac{l}{d} \alpha \sqrt{1+z} \right].$$

(13)

Perform the summation over $l$ and then take the derivative with respect to $\alpha$, arriving to

$$E_{reg}^I = \frac{\pi^2 \hbar c L^2}{2d} \frac{d^2}{d\alpha^2} \frac{d/d\alpha}{\exp[\alpha/d] - 1}. $$

(14)

f) Calculate $U_{reg}$ by taking the limits

$$\lim_{R\to\infty} \lim_{\alpha\to 0} U_{reg}(d, L, R, \alpha)$$

(15)

and show that the energy shift due to the vacuum fluctuations is given by

$$U(d, A) = -\frac{\pi^2 \hbar c A}{720 \, d^3},$$

(16)

where $A = L^2$ is the surface of one plate. Due to this energy shift, there is a non-zero, attractive force between the two plates

$$F = -\frac{\partial U(d, A)}{\partial d} = -\frac{\pi^2 \hbar c A}{240 \, d^4}.$$ 

(17)

Hint: $\frac{y}{e^y-1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} y^n$ where the $B_n$ are the Bernoulli numbers.